Topology Notes

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Lecture 1

Topology

Definition (Topology): A topological space is a set X together with a subset τ so that

- $\varnothing, X \in \tau$
- A finite intersection of elements in τ is in τ
- A arbitrary union of sets in τ is in τ.

The subsets τ are called a topology on X sets in τ are called open sets.

Definition (Basis): Let X be a topological space. A collection of subsets B is called a basis for x if

- Every element in B is open
- Every open set of X is a union of elements in B.

Examples:

- Let (X, d) be a metric space, then d induces a topology on X. Then the sets of elements $B_r(x)$ is a basis.
- The trivial topology. Let X be any set. Then $\tau = \{\varnothing, X\}$ is a topology on X. This is un-interesting but good for counterexamples.
- The discrete topology. Let X be any set. Then $\tau = P(x)$ is a topology on X. This is also good for counterexamples.

Subspace topology

Following are geometrically meaningful topological spaces

Definition (Subspace topology): Let X be a topological space and let S be a subset of X. The subspace topology on S is the topology given by

$$\tau_S = \{U \subseteq S \mid U = S \cap V \text{ such that } V \text{ is some open set in } X\}$$

Examples:

- Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. We topologize S^1 by giving it the subspace topology coming from $\mathbb{R}^2$.
- Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. We topologize S^2 by giving it the subspace topology coming from $\mathbb{R}^3$.
- Consider $[0, \infty)$ in $\mathbb{R}$. Open sets are the type $(a, b) \cap [0, \infty)$. Open sets are of the form $[0, b), b > 0$ or (c, d), $c, d > 0$. Note that $[0, 1)$ is not open in $\mathbb{R}$ but is open in $[0, \infty)$ using the subspace topology.

Definition (Closed sets): A set of C in a topological space X is called closed if X^C is open. Note that sets can be both open or closed, and they are called clopen. Ususally clopen sets are $\varnothing, X$ but others are also possible.

Continuity and homeomorphisms

Proposition:
Finite union of closed sets are closed, arbitrary intersections of closed sets are closed. Proof is using de morgan's law.

Definition (Continuous functions): Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be continuous if whenever O is an open set of Y, then $f^{-1}(O)$ is an open set of X.

Examples:

- Let $X = \{1, 2, 3\}$ with discrete discrete topology. Since every subset is open, every map $f : X \to Y$ is continuous.
- Let $X = \{1, 2, 3\}$ with the trivial topology. Then, a map $f : X \to Y$ is continuous if and only if it is constant. This is because only X is open, so all of its elements are mapped to the same value.

Proposition:
Show that a function $f : X \to Y$ is continuous $\iff$ $f^{-1}(C)$ is closed whenever C is a closed subset of Y.
$\Longrightarrow$: We know that C^c is open, so is $f^{-1}(C^c)$. So $(f^{-1}(C^c))^c$ is closed, which is equal to $f^{-1}(C)$.
$\Longleftarrow$: Let U be open. Then U^c is closed, so is $f^{-1}(U^c) = (f^{-1}(U))^c$. So $f^{-1}(U)$ is open.

Definition (Homeomorphism): A function $f : X \to Y$ is called a homeomorphism if:

- f continuous
- f is a bijection
- f^{-1} is continuous

Two spaces related by a homemorphism are considered the same space, topologically. That is, f let us push the continuous properties of X to Y and f^{-1} allows use to push the continuous properties of Y to X.

Examples:

- Let $\tan(x) : (-\pi/2, \pi/2) \to \mathbb{R}$. tan is continuous, bijection, and arctan is inverse to tan, which is also continuous. This says $(-\pi/2, \pi/2)$ is homemorphic to $\mathbb{R}$. This shows that the size is **not** a topological property.
- Stereographic projection is a homeomorphism between $S^2 \setminus \{pt\}$ and $\mathbb{R}^2$.

Non-examples:

- Let $f : [0, 2\pi] \to S^1$ given by $f(\theta) = (\cos(\theta), \sin(\theta))$. Note that f is continuous, f is a bijection, but f^{-1} is not continuous. To see why, consider the open set $[0, \pi/2)$ in $[0, 2\pi]$, note that $(f^{-1})^{-1}[0, \pi/2)$ is the arc closed at $(0, 0)$ open at $(0, 1)$. This arc is certainly not open in the subspace topology of S^2.

Lecture 2- Quotient spaces and CW complexes

Quotient topology

Note: quotient space captures the intuition of gluing two spaces together. The open sets in the quotient space are the sets that are open in all of the pieces before gluing.

Definition (Quotient topology): Let X be a topologial space and let $\sim$ be an equivalence relation on X. Let $X\backslash\sim = Y$ be the set of equivalence classes of X under $\sim$. We can topologize Y by specifying

$$\tau_Y = \{U \subseteq Y \mid \{x \in X : [x] \in U\} \in \tau_X\}$$

In other words

$$\tau_Y = \left\{U \subseteq Y \mid \left(\bigcup_{u \in U} \{x \in X \mid x \sim u\}\right) \in \tau_X\right\}$$

Proposition:
Let $X = [0, 2\pi]$. Let $\sim$ be the relation $0 \sim 2\pi$. We claim that $Y = X\backslash\{0 \sim 2\pi\} \equiv S^1$. In here, $\equiv$ means homeomorphism.

Proof: Let $f : [0, 2\pi] \to S^1$ be $f(\theta) = (\cos(\theta), \sin(\theta))$. f is bijective (mapping the endpoint to $0 = 2\pi$) and continuous. Let $g : S^1 \to [0, 2\pi]$ be given by $(\cos(\theta), \sin(\theta)) \to \theta$, then g is the continuous inverse of f.

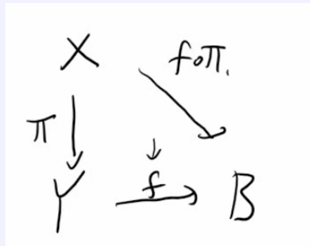
$\square$

Definition (Quotient map (intrinsic definition)): The map $\pi : X \to X\backslash\sim$ given by $x \mapsto [x]$ is continuous, and it is called the quotient map. That is, if $U \subseteq X\backslash\sim$ is open $\iff$ $\pi^{-1}(U)$ is open.

Definition (Quotient map (extrinsic definition)): Let $\pi : X \to Y$ be a surjective map such that $U \subseteq Y$ is open $\iff$ $\pi^{-1}(U)$ is open. Then π is called a quotient map. (I believe the equivalences classes are defined by having the same image under π.)

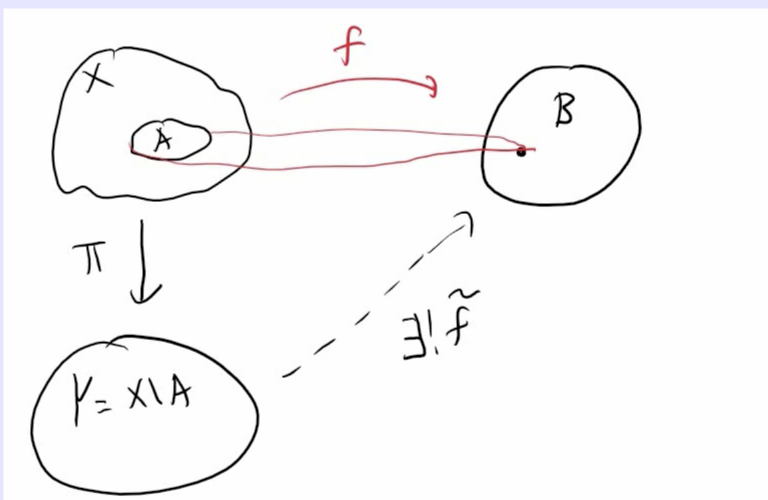
Example Let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be given by $\pi(x_1, \dots x_{n+k}) \to \pi(x_1, \dots, x_n)$, then it is a quotient map.

Theorem (Characteristic prop of quotient topologies):
Let $\pi : X \to Y$ be a quotient map. Then for any topological space B, $f : Y \to B$ is continuous if and only if $f \circ \pi$ is continuous.



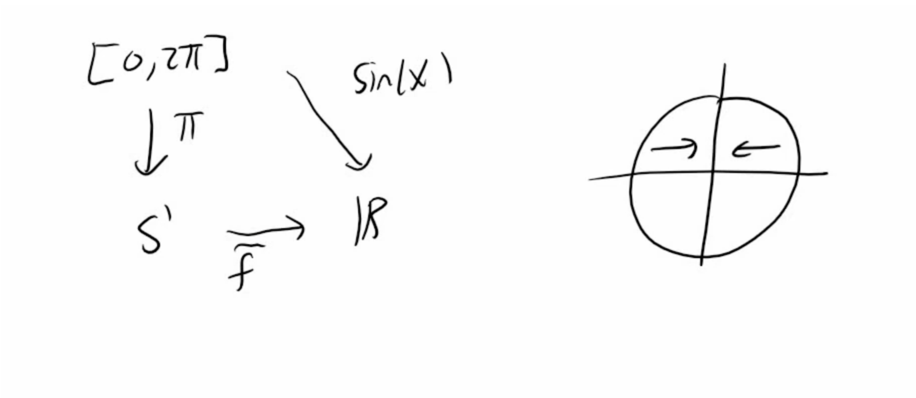
Proof: $\implies$: If f is continuous, then along with the fact that π is continuous, and with the fact that composition of continuous maps are continuous, we know that $f \circ \pi$ is continuous.
$\impliedby$: assume $f \circ \pi$ is continuous. Let $U \subseteq B$ be open. Then, $(f \circ \pi)^{-1}(U) = \pi^{-1} \circ f^{-1}(U)$ is open. By the definition of quotient map, we know that $f^{-1}(U)$ is also open. Therefore f is continuous. $\square$

Corollary (Passing to the quotient): Let $\pi : X \to Y$ be a quotient map and let B be a topological space. Let $f : X \to B$ be a continuous function so that if $\pi(p) = \pi(q)$, then $f(p) = f(q)$. (f maps equivalent elements under the equivalence relations to the same element in B.) Then there exists a unique continuous map $\tilde{f} : Y \to B$ such that $f = \tilde{f} \circ \pi$.



$X \backslash A$ should have been $X \backslash \sim$?

Example: Let $f : [0, 2\pi] \to \mathbb{R}$ be given by $f(x) = \sin(x)$. $\sin(0) = \sin(2\pi)$. This factors to a map $\tilde{f} : S^1 \to \mathbb{R}$ making

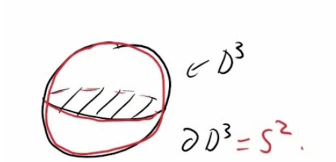


where the factored map is the height of the point on the circle.
The factored map is simply taking any representative and mapping it using f because f is invariant to any representatives you pick in the same equivalence class.

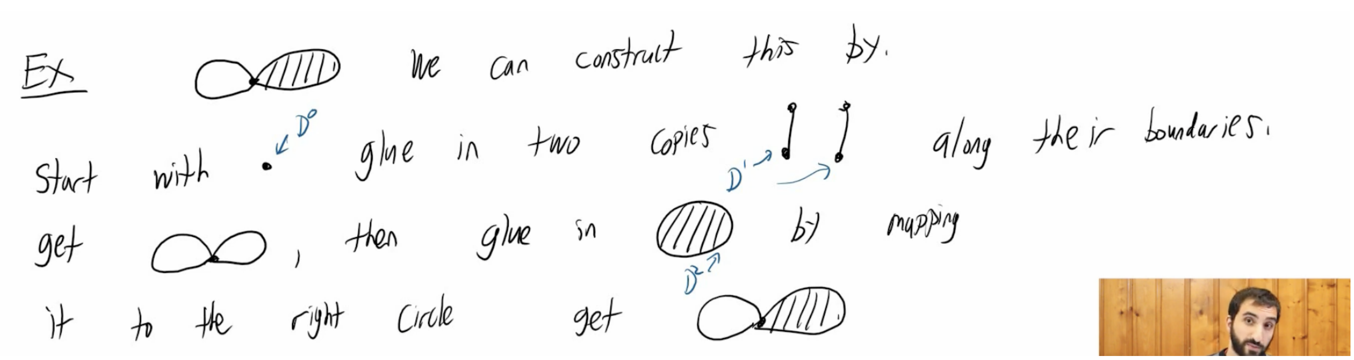
CW complexes

Recall

$$D^n = \{(x_1, \dots, x_n) \subseteq \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$
$$S^{n-1} = \{(x_1, \dots, x_{n-1}) \subseteq \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}$$



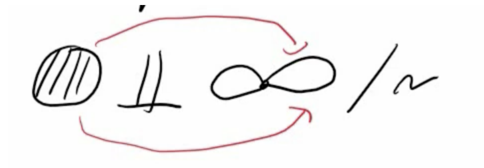
By convention, D^0 is a point.
Example:



Building a CW complex
This process is kind of algorithmic. We build a CW complex via the following procedure

1. Start with a discrete point, X^0, and a bunch of dots.
2. Form the n skeleton X^n from X^{n-1} inductively by attaching D^n to X^{n-1} by a map $\phi : S^{n-1} \to X^{n-1}$. (i.e. Form X^n from $X^{n-1} \coprod D^n$ by the quotient under the equivalence relation $X \sim \phi(x)$.) ($\coprod$ means disjoint union)

3. Either stop at a finite amount of time, setting $X = X^n$ for some n or continue forever giving X the weak topology. A is open if and only if $A \cap X^n$ is open for every n.what is the weak topology? why is this term also in functional analysis?

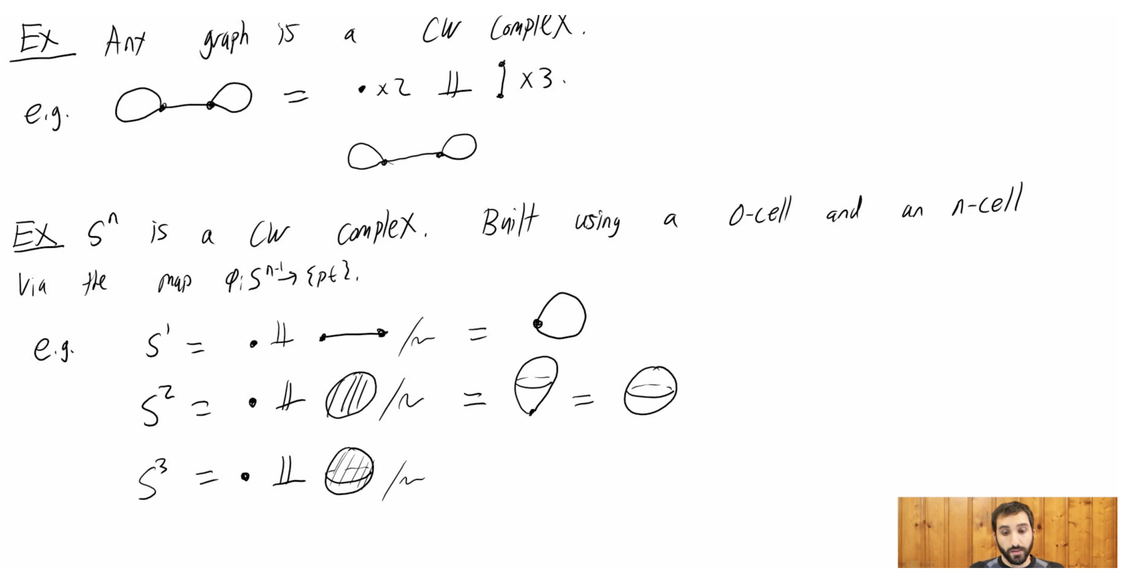


In here, we use the quotient to glue together the boundary of D^2 and X^1. See more DW complexes? Try to construct some of them on your own?

Definition 0.1 (CW complexes):

- If $X = X^n$ for some finite n and $X \neq X^{n-1}$ then we say X is n dimensional.
- X^n is called the n- skeleton. (Or everything that is $\leq n$ dimensional.)
- A closed subset of X that is also a CW complex is called a subcomplex. If A is a subcomplex, we call (X, A) a CW pair.

Here is a list of examples of CW complexes:



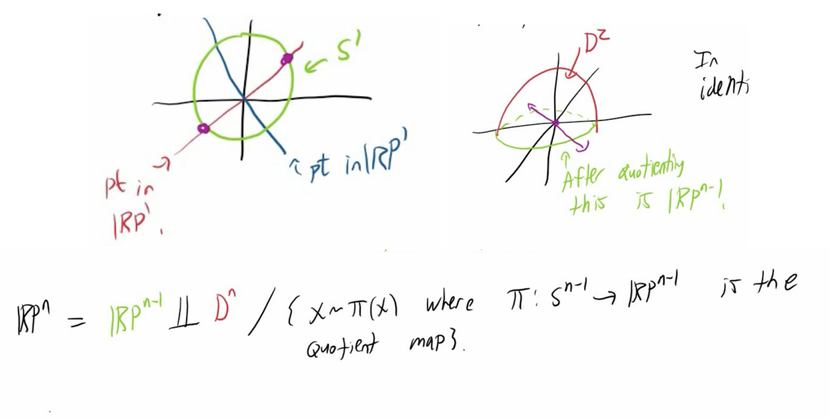
Advanced examples

Consider $\mathbb{R}P^n$. $\mathbb{R}P^n$ is the space of lines in $\mathbb{R}^{n+1}$ that pass through the origin. Note that all points that lie on a single line that pass through origin are in the same equivalence class. Each line in $\mathbb{R}^{n+1}$ hits S^n in two points which are antipodal.
Therefore, we have $\mathbb{R}P^n \equiv S^n \setminus \{x \sim -x\}$., since each point in the upper hemisphere is uniquely associated with a point in the lower hemisphere. So we can restrict ourselves to the upper hemisphere. Note that the upper hemisphere is actually homeomorphic to the space D^n. In the equator, the antopodal points are also

identified. Note that the equator is homeomorphic to $\mathbb{R}P^{n-1}$. Hence, we deduce the relationship

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \coprod D^n / \{x \sim \pi(x), \pi : S^{n-1} \to \mathbb{R}P^{n-1} \text{ is the quotient map}\}$$

That is, we stick the boundaries of the disk to the equator circle, but the equator circle is not just a circle, because $x \sim -x$ is a quotient structure on it, so it's actually $\mathbb{R}P^{n-1}$.



So recursively

$$\mathbb{R}P^n = D^0 \cup D^1 \cup \ldots \cup D^n$$

where D^i is attached to $X^{n-1} = \mathbb{R}P^{n-1}$ by the quotient map.
Note that $\mathbb{R}P^\infty$ is obtained by continuing this process forever.
Can you do this with $\mathbb{R}P^1$? What is D^1?D^0?
Yes, $\mathbb{R}P^0$ is the $\mathbb{R}$ line, and D^1 is a line segment $[0, 1]$. We attach the line segment to $\mathbb{R}$, and obtain the upper semicircle which should represent $\mathbb{R}P^1$? Ask prof how to construct this.

$\mathbb{C}P^n$

Consider the space $\mathbb{C}P^n$. It is the space of complex lines in $\mathbb{C}^{n+1}$. It can be viewed as

$$S^{2n+1}/\{v \sim \lambda v \text{ for } \lambda \in \mathbb{C}, |\lambda| = 1\}$$

A similar analysis from before shows that $\mathbb{C}P^n$ can be built by $D^0 \cup D^2 \cup D^4 \cup \ldots \cup D^{n-2} \cup D^n$ where D^n is attached to X^{n-1} by the quotient map $S^{n-1} \to \mathbb{C}P^{n-2}$.

Problem 1: This is a good exercise to try.

Product topology

Intuitively, the product of $X \times Y$ is a space which has a copy of Y at each point in X. For example $\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2$.

Definition (Product topology): Let X and Y be the topological spaces. The product $X \times Y$ is given as a set by the cartesian product of X and Y. A basis for the topology is $B = \{U \times V \mid U \in \tau_X . V \in \tau_Y\}$.

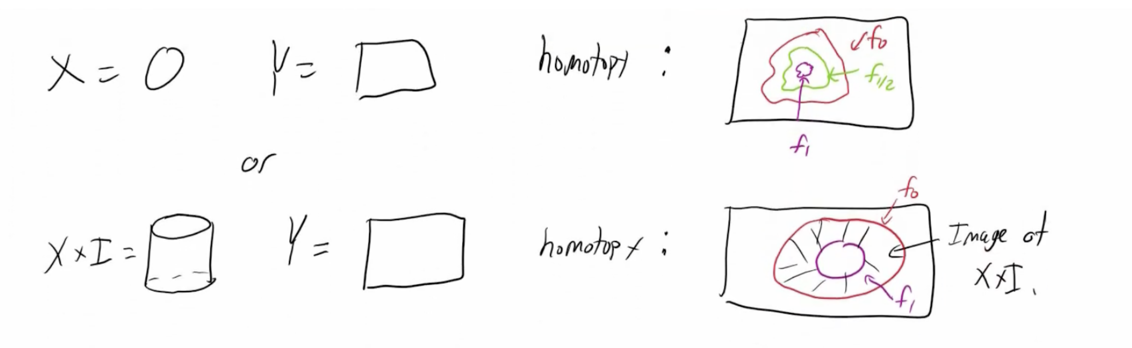
For example, $S^1 \times S^1 \equiv$ torus.

Problem 2: Show that the product of CW complexes is a CW complex, at least think about what the dimension should be.

Homotopy of maps

Definition: Let $g : X \to Y$ and $H : X \to Y$. We say that g and h are homotopic if there exist a family of maps $f_t : X \to Y$, with $t \in [0,1]$ such that

- $f_0 = g$
- $f_1 = f$
- The map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. (note that the topology in the domain depends on the topology of X and I as product topology.)



To think about this pictorally, there are two ways. The first way is to think about the image of X through time, continuously deforming in the space Y. Another way is to think about the image of $X \times I$.
Notation wise, f is homotopic to g, we write $f \simeq g$.

Example 1 Let $h : S^1 \to \mathbb{R}^2$ given by $h(\theta) = (\cos\theta, \sin\theta)$. Let $g : S^1 \to \mathbb{R}^2$ given by $g(\theta) = (0,0)$. Let $f_t(\theta) = ((1-t)\sin\theta, (1-t)\cos\theta)$. Then $f_0 = h$, $f_1 = g$, and f_t is continuous family. So $h \simeq g$. □

Definition 0.2: If a map f is homotopic to a constant map, then f is null-homotopic. One way to think about it is that constant map is a point, so we think of it as it can be shrunk down into a point in Y.

Lecture 3- Homotopy equivalence and the Homotopy Extension Property

Example: (Homotoping id map). Let $I = [0,1]$ and let $f_t : I \to I$ be given by $f_t(x) = (1-t)x$. At $t = 0$, this is the identity map and at $t = 1$, this is the constant map $x \mapsto 0$. We say X morphs into a point.

Homotopy equivalence

Definition 0.3 (Homotopy equivalence): A map $f : X \to Y$ is called a homotopy equivalence if there exists a map $g : Y \to X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$.

Example: the following maps are inverse homotopy equivalences:

- $f : \{pt\} \to [0,1]$ given by $f(pt) = 0$
- $g : [0,1] \to \{pt\}$ given by $g(x) = pt$

We have $f \circ g \simeq id$ because it is homotopic to identity map and $g \circ f \simeq id$ because it's just the point.

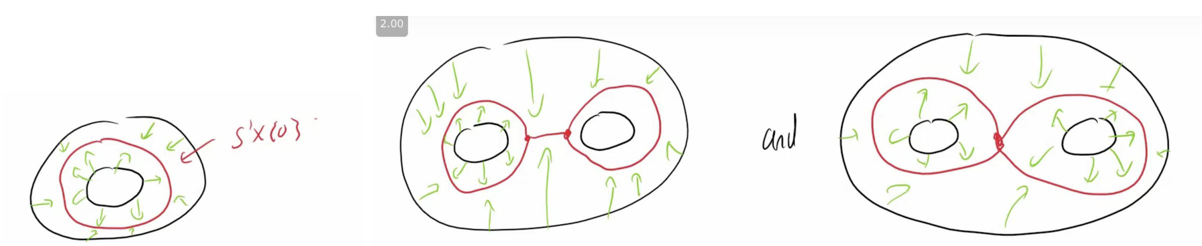
Definition 0.4 (Contractible): A space is called contractible if it is homotopy equivalent to a point. Note that a space having homotopy type of a point is called contractible, which amounts to requiring the identity map of the space to be nullhomotopic.

Definition 0.5 (Deformation retraction): Idea: want a homotopy a space onto a subspace and keep track of the spaces along the way.
A deformation retraction to a space X onto a susbpace A is a homotopy $f_t : X \to X$ so that

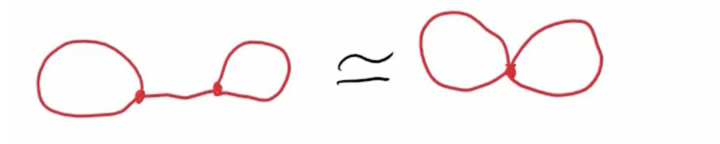
1. $f_0 = id$
2. $f_1(X) = A$
3. $f_t \mid_A = id$ for all t.

Example: Let $S^1 \times [-1,1]$ have coordinates (θ, r). The map $f_t(\theta, r) = (\theta, (1-t)r)$ is a deformation retraction. onto $S^1 \times \{0\}$.

1. $f_0(\theta, r) = (\theta, r)$
2. $f_1(\theta, r) = (\theta, 0)$
3. $f_t \mid_{S^1 \times \{0\}} = (\theta, 0)$



Lemma 0.1 (Lemma 1):
Let A and B be subspaces of X both of which are deformation retraction of X, then A, B are homotopic equivalent.



Definition 0.6 (Retraction): A map $r : X \to X$ is called a retraction of X onto A if $r(X) = A$ and $r\mid_A = id$. Note that a retraction is weaker than a deformation retraction because the latter is continuous and the retraction just straight gives you the end contracted end result instantly. (at time $t = 1$ of a deformation retract.)

<u>Remark</u>
Deformation retraction is a continuous family of continuous function, i.e. an homotopy, while a retraction being just a continuous function.
For example, for any $x_0 \in X$, $\{x_0\} \subset X$ has a retract, pick $r : X \to \{x_0\}$ to be the unique one point map, it is a retract. Yet, $\{x_0\} \subset X$ only has a deformation retraction if X is contractible. Hence, showing a deformation retraction from X onto a subspace A always exhibits X and A are homotopy eqivalent, yet A being a retract of X is weaker!.

Problem 3 (Central problem: homotopy extension problem): Informally: when does a homotopy on $A \subset X$ extend to all of X?

The formal definition is:

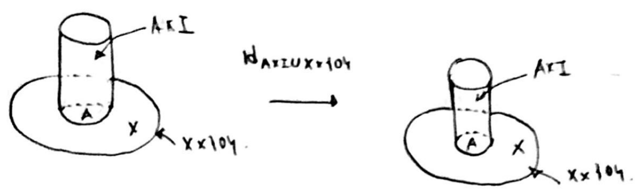
Definition (Homotopy Extension Property (H.E.P.)): A pair (X, A) has the homotopopy extension property, if given a pair of maps $f : X \times \{0\} \to Y$, and $G : A \times I \to Y$, there exists a map $F : X \times I \to Y$ such that $F\mid_{X\times\{0\}} = f$ and $F\mid_{A\times I} = G$.

Another way to understand this property is that suppose one is given a map $f_0 : X \to Y$, and also a homotopy on the subspace $A \subset X$ given by $f_t : A \to Y$ such that it satisfies with $f_0\mid_A$, then we would like to extend to a homotopy $f_t : X \to Y$ given f_0. If given a pair (X, A), such extension problem can always be solved, we say that it has the homotopy equivalence proeprty.

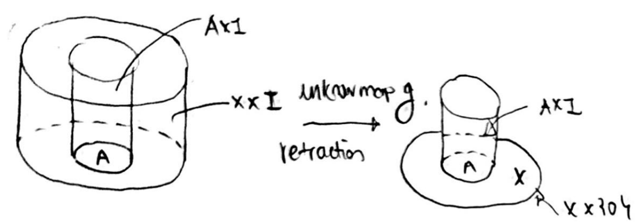
That is, if every pair of maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agrees on $A \times \{0\}$ can be extended to a map $X \times I \to Y$.

Lemma 0.2 (Lemma 2. Pg 14 of Hatcher):
A pair (X, A) has the H.E.P $\iff$ $X \times \{0\} \cup A \times I$ is a retract of $X \times \{I\}$.

Proof:
$\Longrightarrow$
Suppose that (X, A) has the H.E.P. Consider the identity map $I : X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$. It extends to a map $g : X \times I \to X \times \{0\} \cup A \times I$ such that $g\mid_{X\times\{0\}\cup A\times I} = Id$. (its image is the smaller set and it agrees with the identity function on the base set.) Indeed, g is a retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$.



H.E.P. gives:



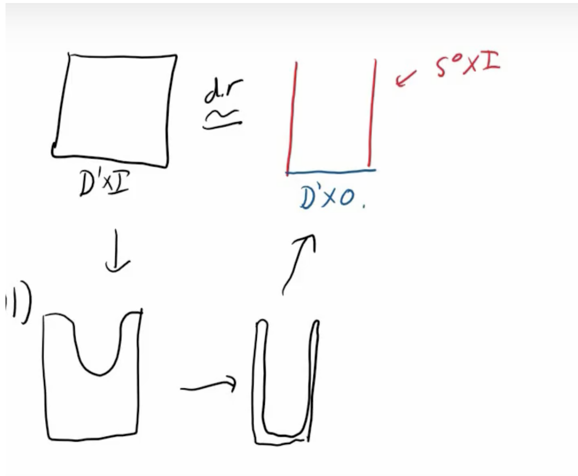
$\Longleftarrow$
Suppose that A is closed. (This assumption is made in order to make claims easier. See textbook to see how to prove it without this assumption.)

Suppose there exists maps $h : A \times I \to Y$, $g : X \times \{0\} \to Y$ such that they agree on $A \times \{0\}$. We define continuous function $f : A \times I \cup X \times \{0\} \to Y$ that agrees with h and g.
Since $A \times I \cup X \times \{0\}$ is a retraction of $X \times I$, we can find a function $k : X \times I \to A \times I \cup X \times \{0\}$ such that it fixes $A \times I \cup X \times \{0\}$. Consider $f \circ k : X \times I \to Y$. This gives a homotopy extension property because $f \circ k$ restricted to $A \times I \cup X \times \{0\}$ gives the same as f. (since k fixes $A \times I \cup X \times \{0\}$.)

□

Theorem 0.3:
If (X, A) is a CW pair, (A is a closed sub-complex of X), then (X, A) has the H.E.P.

Example: $D^n \times I$ retracts onto $D^n \times \{0\} \cup S^{n-1} \times I$.



The lemma showed $D^1 \times I \to D^1 \cup S^0 \times I$. We can also consider $D^2 \times I \to D^2 \cup S^1 \times I$ by retracting from a solid cylinder to the empty toilet paper roll with bottom sealed by a circle.

Lemma 0.4 (Claim 1):
Claim 1
The map $r : D^n \times I \to D^n \times I$ given by

$$r(x,t) = \begin{cases} (\frac{2x}{2-t}, 0) & \text{if } t \leq 2(1 - \|x\|) \\ (\frac{x}{\|x\|}, 2 - \frac{2-t}{\|x\|}) & \text{if } t \geq 2(1 - \|x\|) \end{cases}$$

is a retraction onto $D^n \times \{0\} \cup (S^{n-1}) \times I$. note that x deontes the vector.

Proof:
1. Continuous: matches up when $t = 2(1 - \|x\|)$, and otherwise, its made of continuous functions.
2. Check $r(x, 0) = (x, 0)$ (always in first case).

   Check if $\|x\| = 1$, then $r(x,t) = (x,t)$ (always end up in second case).

This indeed proves it's a retraction. □

Lemma 0.5 (Claim 2):
Claim 2
The map $r(X, t, s) = sr(X, t) + (1 - s)r(X, t)$ is a deformation retraction.
The idea is, you get a retraction first, and the deformation retraction is literally just points moving to the correct position over time $t \in [0, 1]$.

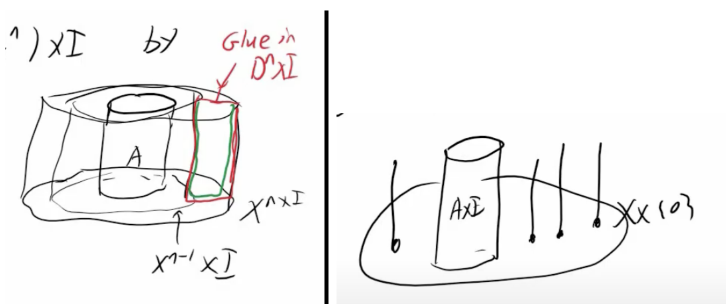
Lemma 0.6 (Claim 3):
Lemma 3
$D^n \times I$ deformation retracts onto $D^n \times \{0\} \cup (S^{n-1} \times I)$.

Proof (Proof of the big theorem):
If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, so (X, A) has the H.E.P.

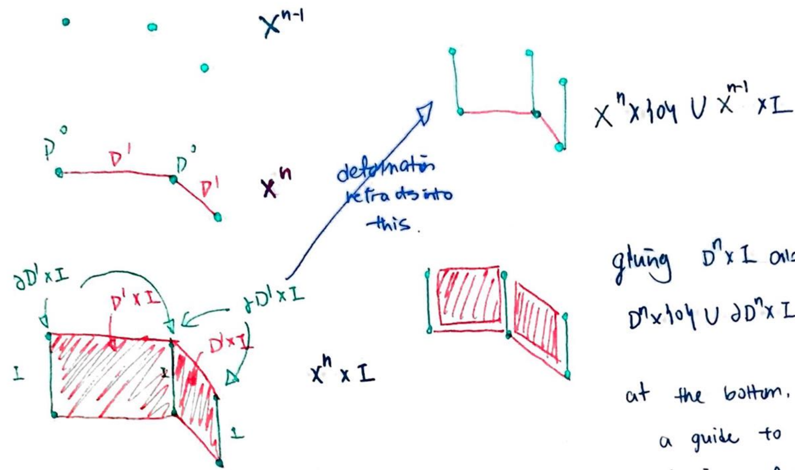
Basic idea is to collapse each n skeleton iteratively (except for A), We make every $X^n \times I$ into $X^{n-1} \times I$ by using the deformation retract of $D^n \times I$ into $D^n \times 0 \cup \partial D^n \times I$. So we still keep the set $X^n \times \{0\}$ (the big set times 0 is always kept), but the big set times I would be collapsed.

Then. when we get 0, we know what to do.

So, $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^{n-1} \times I$, but by lemma 3, $D^n \times I$ deformation retracts onto $D^n \times \{0\} \cup \partial D^n \times I$.
So $X^n \times I \cup X \times I$ retracts onto $(X^{n-1}) \times I \cup (A \times I) \cup (X^n \times 0)$ (That is, A we leave unchanged, and we collapse $X^n \times I$ to $X^{n-1} \times I \cup X^n \times \{0\}$). We try to collapse every that is not in A into the lower dimension skeleton.
Inductively we may repeat this process with each cell in each skeleton to get a deformation retraction $X^n \times I \cup A \times I$ to $X \times \{0\} \cup X^0 \times I \cup A \times I$. Now we can squish down the 0 cells (the bars into bottom points) to $X \times \{0\}$, to get a retraction onto $X \times \{0\} \cup A \times I$. So, since it is a reformation retraction, we have X, A satisfies the H.E.P. $\square$



The following illustration is a quite accurate one. You can also draw it as if D^1 is a loop.



gluing $D^n \times I$ along each of.
$D^n \times \{0\} \cup \partial D^n \times I$.

at the bottom. $D^n \times \{0\}$ gives a guide to glue in the D^n - dimension of $D^n \times I$. Then, we glue $D^n \times I$ to its boundary $\partial D^n \times I$ as the I dimension match up and the "bottom" $D^n \times \{0\}$ and $\partial D^n \times \{0\}$ match up.

Proposition 0.7:
If (X, A) satisfies the H.E.P. and A is contractible, then $X/A \simeq (h.e.)X$. In fact, the quotient map q is a h.e.

Proof: Since A is contractible, there exists a homotopy $C_t : A \to A$ contracting A. That is, $C_0 = id, C_1(A) \mapsto pt$. Since X, A has the H.E.P. we can make a map f_t such that $F(X, \{0\})$ is identity and $F(x, t) = C_t(x)$ for $x \in A, t \in [0, 1]$. We extend C_t to maps $f_t : X \to X$ such that it is equal restricting to C_t, $f_t\mid_A = C_t$.
Since $f_t(A) \subseteq A$, (because f_t extends C_t, and that $C_t(A) \subseteq A$ at all times) the characteristics property of quotient maps says that there exists a map $\overline{f_t} : X/A \to X/A$ making the following commute. (We are using "passing to the quotient here". $q \circ f_t : X \to X/A$ is a continuous function and $q : X \to X/A$ is a quotient. The passing to the quotient implies there exists a function $\overline{f}_t : X/A \to X/A$ such that $\overline{f}_t \circ q$ is equal to the original function, which is $q \circ f_t$.) So $q \circ f_t = \overline{f}_t \circ q$.

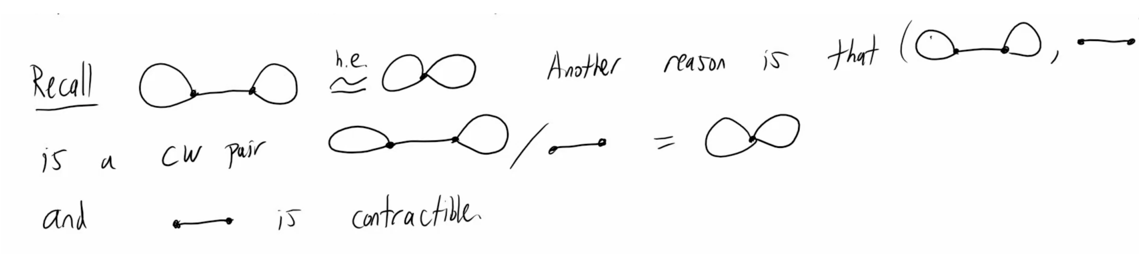
$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_t"'] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "g"] \arrow[r, "\overline{f}_1"'] & X/A \end{tikzcd}$$

The following is important: (Now this refers to the image on the right.)
But since $f_1(A) = pt$ (lands in a point), we get a map $g : X/A \to X$. The equivalence relation X/A is that $x \sim y \iff x, y \in A$. So, if $q(x) = q(y)$, (that is, they are both in A, so in the same equivalence class) then

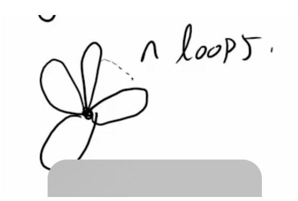
$f_1(x) = f_1(y)$ because they are the same point that A is mapped to. So g can be defined this way, by the "passing to the quotient".

Moreover, g commutes with everything in the diagram. That is, $q \circ g = \overline{f_1}$ and $g \circ q = f_1$.
Now, $f_1 \simeq id_X$ by f_t and $\overline{f_1} \simeq id_{X/A}$ by $\overline{f_t}$. That is, the homotopy is given by t ranging from 0 to 1, as $\overline{f}_0, f_0$ are both identity on their respective domains.
So q, g are inverse homotopic equivalent. Particularly, X and X/A are homotopic equivalent.
Reference: Hatcher page 16

$\square$



Generalizing this, every finite graph is homotopy equivalent to a wedge of n circles. (flower shape.)



The idea is that if an edge joins two distinct vertices, then the edge is contractible. Quotienting out by all of those edges leave only edges whose start and end points are the same.

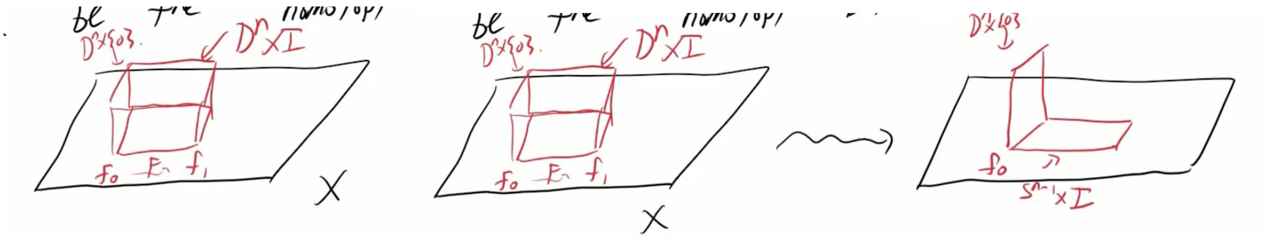
Attaching maps only depend on homotopy class

There is another way to show homotopic equivalence without finding the explicit homotopy equivalence functions.

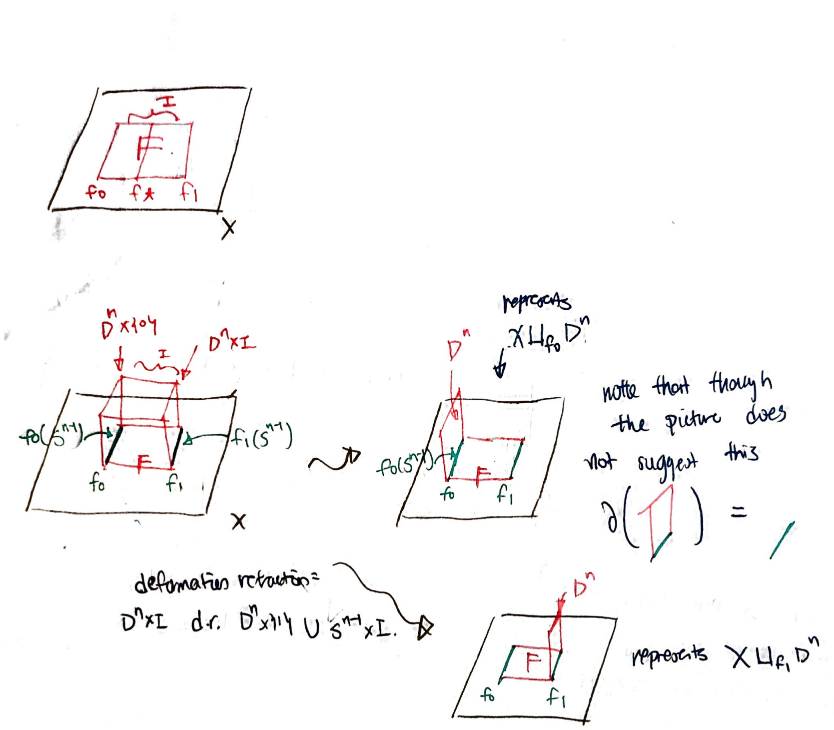
Definition 0.7 (Attaching spaces): Suppose we have space X_0, X_1, and we wish to attach X_1 to X_0 by identifying the points in a subspace $A \subset X_1$ with points in X_0. So we have a map $f : A \to X_0$, and we form a quotient space $X_0 \sqcup X_1$ by identifying each point $a \in A \subset X_1$ with its image in $f(a) \in X_0$. So we denote this quotient space by $X_0 \sqcup_f X_1$, the space X_0 with X_1 attached along A via f. When $(X_1, A) = (D^n, S^{n-1})$, we have the case of attaching an n cell to X_0 via a map $f : S^{n-1} \to X_0$.

Theorem 0.8:
Let X be a CW complex. Let D^n be the n disk. Let $f_0, f_1 : S^{n-1} \to X$ be two homotopic maps. Then $X \sqcup_{f_0} D^n \simeq X \sqcup_{f_1} D^n$.

Proof: Let $F : S^{n-1} \times I \to X$ be the homotopy between f_0 and f_1. Let $B = X \bigcup_F D^n \times I$.
By previous lemma, $D^n \times I$ deformation retracts onto $D^n \times \{0\} \cup S^{n-1} \times I$.



Doing the d.r. internally shows that $B \simeq X \sqcup_{f_0} D^n$ (d.r.)
Flipping the d.r. upside down(deformation retracting $D \times I$ towards 1 instead of the 0 direction),gives $D^n \times I$ d.r. into $D^n \times \{0\} \cup S^{n-1} \times I$. This gives that $B \simeq X \sqcup_{f_1} D^n$ (d.r.). The two spaces $X \sqcup_{f_0} D^n, X \sqcup_{f_1} D^n$ are both deformation retracts of the big space B, so they are homotopy equivalent. $\square$



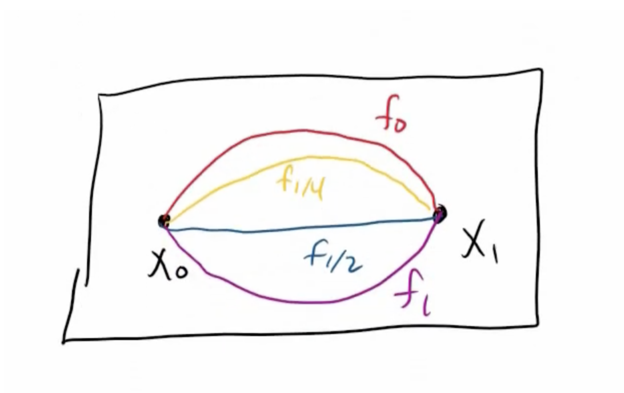
Lecture 4. Paths and Fundamental Groups

Definition 0.8 (Path): A path in a space X is a map $f : I \to X$.

Definition 0.9 (Path connected): A path in a space X is path connected if any two points are joined by a path.
e.g. $\mathbb{R}^n$ is joined by a line. $(0,1) \cup (2,3) \subset R$ is not path connected from the intermediate value theorem.

Definition 0.10 (Homotopy of paths in X): A homotopy of paths in X is a family of paths $f_t : I \to X$ such that

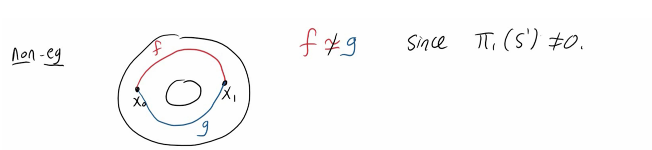
1. $\forall t \in [0,1]$, $f_t(0) = x_0$, $f_t(1) = x_1$ for some fixed x_0 and x_1 in X. That is, the homotopy of paths always start and end at the same endpoints.
2. $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.



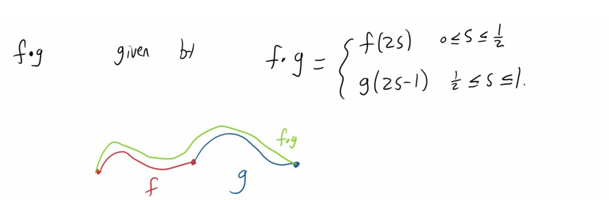
For example, if $D \subset \mathbb{R}^n$ is convex domain, then any two paths f_0, f_1 with same endpoints are homotopic. that is

$$f_t(s) = t f_1(s) + (1-t) f_0(s)$$

An non-example is two paths in an annulus.



Given paths f, g with $f(1) = g(0)$, there is a path product



You basically travel twice as fast. Path products respect homotopy of paths, that is, you can define products on the homotopy of paths.

Lemma 0.9 (Product on homotopy of paths):
If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

Proof: Let

$$H_t(s) = \begin{cases} f_t(2s) & = 0 \leq s \leq \frac{1}{2} \\ g_t(2s-1) & = \frac{1}{2} \leq s \leq 1 \end{cases}$$

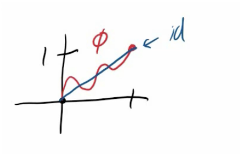
So if $[f]$ is the equivalence class of all paths homotopic to f and $[g]$ is the equivalence class of all paths homotopic to g then $[f] \cdot [g]$ is well defined as $[f \cdot g]$. $\square$

Definition (Loop): A loop is a path $f : I \to X$ so that $f(0) = f(1)$..

Example 2 For example, since $I/\{0 \sim 1\} \cong S^1$, a loop is the same thing as a map $S^1 \to X$. $\square$

Definition (Fundamental group): The set (actually a group) of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint $x_0 (f(0) = f(1) = x_0)$ is called the fundamental group of X and is denoted $\pi_1(X, x_0.)$

Definition (Reparametrization): Let $f : I \to X$ be a path and let $\varphi : I \to I$ be a map with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then the path $f \circ \varphi$ is called a reparametrization of f.



Proposition 0.10:
If $f : I \to X$ is a path and $f \circ \varphi$ is a reparametrization, then $f \circ \varphi \simeq f$.

Proof: Since I is convex, there exist a homotopy φ_t from φ to id given by

$$\varphi_t(s) = (1-t)\phi(s) + ts$$

so $f \circ \varphi_t$ is a homotopy between $f \circ \varphi$ and f. $\square$

So all those $\varphi : I \times I \to I$ maps are homotopic to the identity and all reparametrizations are homotopic to the original map f.

> **Theorem 0.11 (Theorem (Fundamental group)):**
> $\pi_1(X, x_0)$ is a group under $\cdot$.

Proof: We need identity, inverses, and associativity.
Identity:
Claim: constant path $e : I \to X$ given by $e(s) = x_0$ is an identity under $\cdot$.
If $f : I \to X$ is a loop:

$$f \cdot e = \begin{cases} f(2t) = 0 & \text{for } 0 \leq s \leq \frac{1}{2} \\ x_0 & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

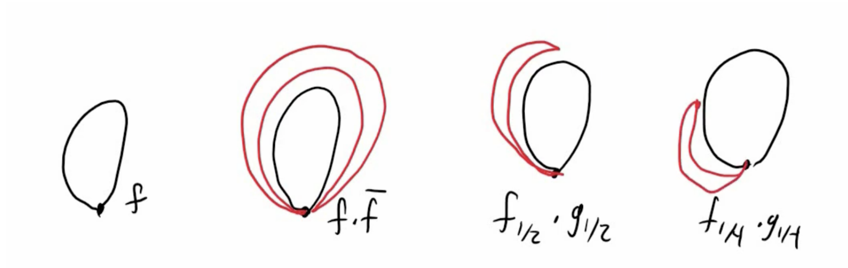
This is a reparametrization of f under φ :

$$\varphi = \begin{cases} 2t & \text{for } 0 \leq s \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since all reparametrizations are homotopic, $f \cdot e \simeq f \circ \varphi \simeq f$.
Similarly, $e \cdot f \simeq f$.
Inverses
Let $f : I \to X$ be a loop and let $\overline{f} : I \to X$ be a loop defined by $\overline{f} = f(1-s)$. (Runs f in reverse)
Claim: $f \circ \overline{f} \simeq \overline{f} \circ f \simeq e$.
Let f_t be given by

$$f_t = \begin{cases} f & \text{on } [0, 1-t] \\ f(1-t) & \text{on } [1-t, 1] \end{cases}$$

and let $g_t = \overline{f_t}$.
Now construct homotopy.
Let $h_t = f_t \cdot g_t$. Then $h_1 = e$. $h_0 = f_0 \cdot g_0 = f \cdot \overline{f}$.
This is a homotopy from $f \cdot \overline{f} \to e$.



Similarly, $\overline{f} \cdot f \simeq id$ so $\overline{f} = f^{-1}$.
Associativity
Need to show that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$.

$$(f \cdot g) \cdot h = \begin{cases} \text{Do } f \text{ 4 times as fast} \\ \text{Do } g \text{ 4 times as fast} \\ \text{Do } h \text{ 2 times as fast} \end{cases}$$

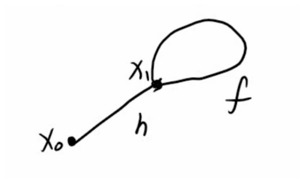
$$f \cdot (g \cdot h) = \begin{cases} \text{Do } f \text{ 2 times as fast} \\ \text{Do } g \text{ 4 times as fast} \\ \text{Do } h \text{ 4 times as fast} \end{cases}$$

These are reparametrizations of the same path so they are homotopic. So, $\pi_1(X, x_0)$ is indeed a group.

□

Example 3 $\pi_1(\mathbb{R}^n, 0)$ is trivial. Let $C_t : \mathbb{R} \to \mathbb{R}$ be the contraction $x \mapsto (1-t)x$. Let $f : I \to \mathbb{R}^n$ be a loop based at 0. Then $C_t \circ f$ is a homotopy of g onto the trivial path. More generally, if C is any contractible space which deformation retracts onto X_0 then $\pi_1(C, x_0) = \{id\}$. □

A natural question is how much does the fundamental group depend on its basepoint?
Let f be a loop based at x_1 and let h be a path from x_0 to x_1.



We get a loop based at x_0 given by $h \cdot f \cdot \overline{h}$. We define the change of basepoint map B_h to be

$$B_h : \pi_1(X, x_0) \to \pi_1(X, x_1), f \mapsto h \cdot f \cdot \overline{h}$$

Proposition 0.12 (Change of basepoint map is an isomorphism):
The map B_h (if the basepoints are path connected) is an isomorphism.

Proof: B_h is a homomorphism since $B_h(f \cdot g) = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h}] \cdot [h \cdot g \cdot \overline{h}] = B_h(f) \cdot B_h(g)$
Morevoer, B_h has a inverse since $B_h \cdot B_{\overline{h}}(f) = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$. So B_h is an isomorphism.

□

Note that if X is path connected, $\pi_1(X, x_0)$ is independent of x_0 so we can just write $\pi_1(X)$. But if not, we cannot write it this way.

Induced homomorphisms

Definition 0.11 (Pointed space and pointed maps): If X is a space and x_0 is a pt in X, we call (X, x_0) a pointed space. If (X, x_0) and (Y, y_0) are two pointed spaces, we call $\varphi : (X, x_0) \to (Y, y_0)$ a pointed map if $\varphi(x_0) = y_0$. (The notation implies this is a pointed map.)
If $f : I \to X$ is a loop based at x_0, and $\varphi : (X, x_0) \to (Y, y_0)$ is a pointed map, then we get loop based at y_0 by $\phi \cdot f$. This respects homotopies of paths, so we get a well defined map

$$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), [f] \mapsto [\varphi \circ f]$$

Also, $\varphi(f \cdot g) \simeq \varphi(f) \cdot \varphi(g)$, so φ_* is a homomorphism.

Basically, ψ, φ are maps from one space to another while ψ_*, φ_* maps the elements from the fundamental group to the fundamental group of another space.

More properties:

- If $\varphi : (X, x_0) \to (X, x_0)$ is the identity map, then φ_* is $id_{\pi_1(X,x_0)}$.
- If $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, then

$$(\psi \circ \varphi)_* : \pi_1(X, x_0) \to \pi_1(Z, z_0) = \psi_* \circ \varphi_*$$

  so that $\pi_1(X)$ is a functor.

Definition 0.12 (Pointed homotopy equivalent): Let (X, x_0) and (Y, y_0) be pointed spaces. We say (X, x_0), (Y, y_0) are pointed homotopy equivalent, if there exists

$$\varphi : (X, x_0) \to (Y, y_0) \text{ and } \psi : (Y, y_0) \to (X, x_0)$$

such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity map by a homotopy fixing x_0 (or y_0) at all times. That is, it's almost the same as homotopy equivalent, except that throughout the time, the basepoints are fixed.

Proposition 0.13:
If (X, x_0) and (Y, y_0) are pointed homotopy equivalent, then

$$\pi_1(X, x_0) \simeq (isom)\pi_1(Y, y_0)$$

Proof: Let $f : I \to X$ be a loop and let $\varphi : X \to Y, \psi : Y \to X$ be inverse pointed homotopy equivalences. $\psi \circ \varphi \circ f : I \to X$ is a loop in X based at x_0. Let $(\psi \circ \varphi)_t$ be the basepoint fixing homotopy to id. So $(\psi \circ \varphi)_0 = \psi \circ \varphi$, $(\psi \circ \varphi)_1 = id_X$. Then $(\psi \circ \varphi)_t \circ f$ is a homotopy from $\psi \circ \varphi \circ f$ and f. since $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$, φ_* and ψ_* are isomorphisms. $\square$

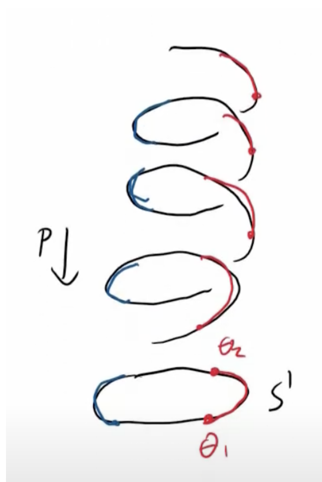
Remark: We can drop the conditions on basepoints with more work. For now, we can state that homotopy equivalences induce isomorphism.

Lecture 5. Covering spaces and the Fundamental group of the circle.

Idea is that in a covering space, the bottom spaces are harder to deal with, but the pancakes above would be easier to deal with. What we do is that we lift the loops in the bottom space to paths in the top space, which will be much easier to deal with.
Goal: show that $\pi_2(S^1) \cong \mathbb{Z}$ with the isomorphism given by

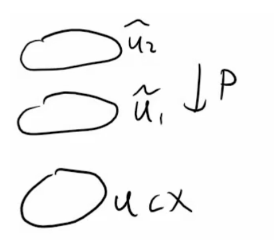
$$n \mapsto (\cos(2\pi ns), \sin(2\pi ns)) \; 0 \leq s \leq 1.$$

To do this, we are going to "lift" loops to paths in $\mathbb{R}$. Consider $\mathbb{R} \hookrightarrow \mathbb{R}^3$ given by $s \mapsto (\cos(2\pi s), \sin(2\pi s), s)$. Also in $\mathbb{R}^3$, is S^1 given by $(\cos(2\pi s), \sin(2\pi s), 0)$ for $0 \leq s \leq 1$. We get a map $p : \mathbb{R} \to S^1$ by projecting onto the first two factors.
To show this map is open, we note that $p^{-1}(\theta_1, \theta_2)$ is a disjoint union of open sets. We imagine the covering space as stack of pancakes.

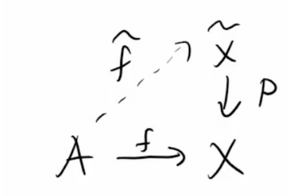


Covering spaces

Definition 0.13 (Covering space): Given a space X, a covering space of X, denote by $\tilde{X}$, is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$, such that for every $x \in X$, there is an open set U, $x \in U$, so that $p^{-1}(U)$ is a disjoint union of open sets $\tilde{U_i}$, eahc of which projects onto U homeomorphically via P. That is, $P \mid_{\tilde{U_i}}: \tilde{U_i} \to U$ is a homeomorphism. Each set U here is called "evenly covered" by P.



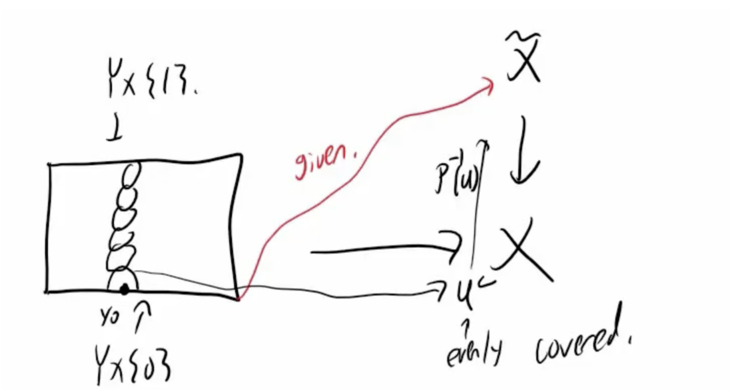
Definition 0.14 (Lift): Let $f : A \to X$ be a map. A map $\tilde{f} : A \to \tilde{X}$ for some covering space $P : \tilde{X} \to X$ is called a lift of f if $p \circ \tilde{f} = f$.



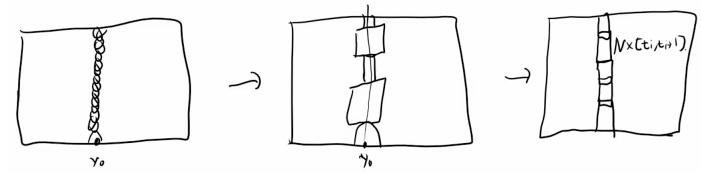
We will now show that covering spaces satisfy the homotopy lifting property:

Proposition (Homotopy lifting property):
Let X be a topological space and let $P : \tilde{X} \to X$ be a covering space. Let $F : Y \times I \to X$ be a map and let $\tilde{F} : Y \times \{0\} \to \tilde{X}$ be a lift of $F\mid_{Y \times \{0\}}$. Then there exists a unique map $\tilde{F} : Y \times I \to \tilde{X}$ lifting F and restricting to the given lift on $Y \times \{0\}$.

The proof idea is we first have y_0 on $Y \times \{0\}$. Take small neighbourhood around y_0, transfer it to X such that it is homeomorphic to some space that is in $\tilde{X}$. We can choose the one upstairs such that it contains $\tilde{y_0}$, as it is already defined. Then we "tile" the $Y \times I$ space this way.

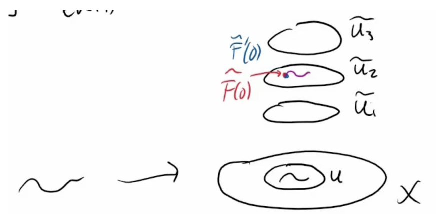


Proof: Let $y_0 \in Y$. We first construct a lift on $N \times I$ for some open neighbourhood N, $y_0 \in N$. Since $F : Y \times I \to X$ is continuous and $Y \times I$ has the product topology, each point in $y_0 \times I$ has a neighbourhood $N_t \times (a_t, b_t)$, so that $F(n_t \times (a_t, b_t))$ is evenly covered. Since I is compact (Heine Borel), a finite number of such sets covers $y_0 \times I$. Moreover, intersecting all of the N_t remaining gives a single neighbourhood of Y called N. (equivalent to taking the narrowest square. cuz finite intersection of open sets is open.) N is the intersection of the n_ts, which still cover the point y_0. Now, we have neighbourhoods $N \times [t_0, t_1] \cup N \times [t_1, t_2] \cup \ldots \cup N \times [t_{m-1}, t_m]$ with $t_0 = 0, t_m = 1$, covering $y_0 \times I$ and evenly covered.



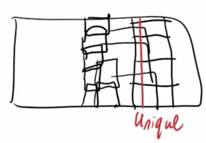
We next construct the lift on $N \times [0,t_1]$. $F(N \times [0,t_1])$ is evenly covered so we can declare $\tilde{F}(N \times [0,t_1])$ to be $p^{-1}F(N \times [0,t_1])$. (i.e. one of the stacking pancakes upstairs). Since $\tilde{F}(N \times \{0\})$ is given, we have a preferred $\tilde{U_i}$. Define $\tilde{F}(N \times [0,t_i])$ to be $P^{-1} |_{\tilde{U_i}} F(N \times [0,t_1])$, where $\tilde{U_i}$ is the component continuing the given lift of $N \times \{0\}$.
We may proceed inductively to construct the lift on all of $N \times I$.

<u>Detour</u>. We will prove that a lift is unique in the case that Y is a point. So what we are doing, is to lift an interval. Let $F : I \to X$ be a map. Suppose $\tilde{F}, \tilde{F}'$ are two lifts which agree on 0. As before, partition I into evenly covered neighbourhoods, $[0,t_1], [t_1,t_2], \dots [t_{m-1},t_m]$. Since $[0,t_1]$ is connected, so is $\tilde{F}[0,t_1]$ and $\tilde{F}'[0,t_1]$. Since $[0,t_1]$ is evenly covered, it lands in a set $U \subset X$ covered by $\tilde{U_i} \subset \tilde{X}$.
Since $\tilde{F}(0) = \tilde{F}'(0)$, the image of $[0,t_1]$ must land in the same $\tilde{U_i}$.



Since p is a homeomorphism on $\tilde{U_i}$, the lifts must completely agree. Proceed from here inductively.

Since we have constructed $\tilde{F}$ on a neighbourhood of each point $y \in Y$, and we have shown the restriction to each line is unique. We can fetch together the constructed lifts uniquely to get the desired lift $\tilde{F} : Y \times I \to \tilde{X}$.



$\square$

Corollary 1: For each path $f : I \to X$ starting at $x_0 \in X$ and each $\tilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x_0}$.

Corollary 2: Let $p : \tilde{X} \to X$ be a covering space and let $f_t : I \to X$ be a homotopy of paths starting at x_0. Let $\tilde{x_0} \in p^{-1}(x_0)$. Then there is a unique lifted homotopy $\tilde{f_t} : I \to \tilde{X}$ of paths starting at $\tilde{x_0}$.

Proof: Let $F : I \times I \to X$ be given by $F(s,t) = f_t(s)$. By Cor 1, we get a unique lift of $F(I \times \{0\})$. By proposition previously, we get a unique lift $\tilde{F} : I \times I \to \tilde{X}$. Since $f_t(0) = F(0,t) = x_0$ and $f_t(1) = F(1,t) = x_1$, so $\tilde{F}(\{0\} \times I)$ and $\tilde{F}(\{1\} \times I)$ are constant. So $\tilde{f}_t(s) = \tilde{F}(s,t)$ is a homotopy of paths. □

0.1 Fundamental group of S^1

Recall that $P : \mathbb{R} \to S^1$ given by $s \mapsto (\cos(2\pi s), \sin(2\pi s))$ is a covering map. Notation: Let ω_n be the loop given by $(\cos(2\pi s), \sin(2\pi s))$, $0 \leq s \leq 1$.

Theorem:
$\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$.

Proof: Let $f : I \to S^1$ be a loop based at $(1,0)$. By Cor 1, there is a unique lift of f starting at 0. Since $P^{-1}(1,0) = \mathbb{Z}$, $\tilde{f}(1)$ is an integer n.
Another path which starts at 0 and ends at n is $\tilde{\omega_n}$. There is a homotopy in $\mathbb{R}$ between $\tilde{f}$ and $\tilde{\omega_n}$ given by

$$(1-t)\tilde{f} + t\tilde{\omega_n}$$

Composing $\tilde{f}_t$ with p gives a homotopy from f to ω_n.
Suppose that $f \simeq \omega_n$ and $f \simeq \omega_m$. Let f_t be the homotopy between them. By corollary 2, there exists a lifted homotopy $\tilde{f}_t$ from $\tilde{\omega_n}$ to $\tilde{\omega_m}$. But $\tilde{f}_t$ is a homotopy of paths, so the endpoints are fixed.
In particular,

$$n = \tilde{f}_0(1) = \tilde{f}_1(1) = m$$

so $n = m$. So $\pi_1(S^1) \simeq \mathbb{Z}$. Revisit this proof if you want? This is interesting and important. I know this works and I understand it. But i cannot recite the key components to making this proof work. So I should revisit the proof. □

Lecture 6. Applications of fundamental group of the circle

Theorem:
Every continuous map $f : D^2 \to D^2$ has a fixed point. That is, $f(x_0) = x_0$ for some $x_0 \in D^2$.

Proof: Suppose for all $x \in D^2$, $f(x) \neq x$. We can define $r : D^2 \to S^1$ as follows: draw a ray from $f(x)$ to x and declare $r(x)$ to be the point on S^1 where this ray hits S^1.
Note that r is continuous since all small change in x produces a small change in $f(x)$, which produces a small change in $r(x)$.
Also note that $r\mid_{S^1} = id$. So r is a retraction. Recall **fact** that D^2 is simply connected since it is contractible. So any loop on D^2 is homotopic to a constant loop. Let f be any loop in $S^1 \subset D^2$. Let f_t (in D^2) bea homotopy of f to a trivial loop.

Then consider $r \circ f_t$. This is now a homotopy in S^1. But f_1 is constant, so $r \circ f_t$ is in fact a homotopy of f to a trivial loop. But not all loops in S^1 are trivial in $\pi_1(S^1)$. This is a contradiction.

□

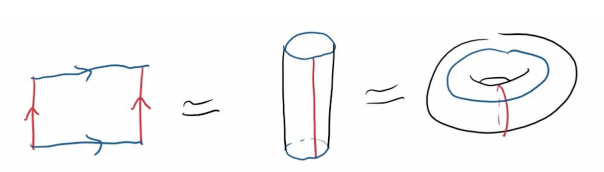
Fundamental group of n-dimensional tori

Proposition:
If X and Y are path connected spaces then $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

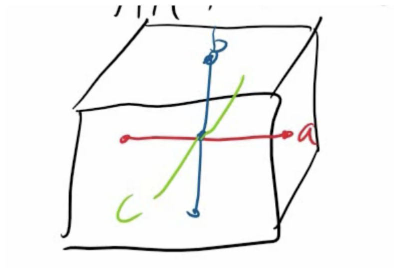
Proof: map $f : Z \to X \times Y$ is continuous if and only if $f(z) = (g(z), h(z))$, where $g : Z \to X$ and $h : Z \to Y$ are continuous. Let $f : I \to X \times Y$ be a loop based at (x_0, y_0). We get a loop $g \in X$ based at x_0 by looking at the first vector. We also get a loop h in Y based in y_0. Moreover, a homotopy f_t of f gives two homotopies g_t and h_t in X and Y respectively.
We get a map $P : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g], [h])$. This map is bijective and a homomorphism so there groups are isomorphic. □

Definition 0.15 (n diml torus): The n dimensional torus is the space $T^n = \underbrace{S^1 \times S^1 \times \ldots S^1}_{n \text{ times}}$. For example, $T^2 = S^1 \times S^1 = \mathbb{Z} \times \mathbb{Z}$. Note that if a and b generate $\pi_1(X)$ and $\pi_1(Y)$ respectively, then (a, y_0) and (x_0, b) generate $\pi_1(X \times Y)$.

Recall that T^1 is $S^1 \times S^1$ and S^1 is $[0, 1]/\{0 \sim 1\}$. So T^2 can be obtained by the following:



Similarly, $T^3 \simeq I/\{0 \sim 1\} \times I/\{0 \sim 1\} \times I/\{0 \sim 1\}$. So $\pi_1(T^3) = \mathbb{Z}^3$. So $\pi_1(T^3)$ is generated by a, b, c, where those are loops in the x, y, z directions respectively, that come in at one vector, go out of the cube, and rejoin the cube at the center of opposite size, and complete the loop. An image illustration is:



In general, $\pi_1(T^n) = \mathbb{Z}^n$.

Fundamental theorem of algebra

Theorem (Fundamental theorem of algebra):
Every non-constant polynomial in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: By dividing by the leading term we get a monic polynomial:

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$$

If $p(z)$ has no roots in $\mathbb{C}$, then we can define for each $r \in \mathbb{R}$, a continuous function $f_r : I \to S^1$ by

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

Note that $f_r(0) = f_r(1)$. So this is a loop in S^1. At $r = 0$, this is the constant loop, so $f_r(s)$ defines a null homotopy.
Now, choose a very large r. Larger than 1, also larger than $|a_0| + |a_1| + \ldots + |a_{n-1}|$. Now, for $|z| = r$, we get that

$$|z^n| = r \cdot r^{n-1} > (|a_0| + |a_1| + \ldots + |a_{n-1}|) \cdot |z|^{n-1}$$

since $r > 1$, $|z| > 1$, so $|z|^n > |a_0 + a_1 z + \ldots + a_{n-1}z^{n-1}|$. Let $p_t = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)$ be a family of polynomials.
Recall that

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

Plugging in p_t into the above formula gives a homotopy between $f_r(s)$ (at $t = 1$) to $\omega_n(s) = e^{2\pi ins}$, (when $t = 0$.)
So, letting r go from 0 to 1 gives a homotopy form trivial loop to $f_1(s)$ but letting t go to 0 takes this loop to $\omega_n(s)$. This means $\omega_n(s)$ is homotopic to a constant loop. This is a contradiction unless $n = 0$, which is the constant polynomial. This means all non constant polynomial has a root.

$\square$

Revisit the above proof, though proof makes sense by the $/p(x)$ in the denominator and numerator seems unnecessary??

Lecture 7

0.2 Free Products

Suppose we are give groups G and H, and we want to form a group containing both spaces. One way is to form $G \times H$, but here if we have $g_1 \in G$, $h_1 \in H$, then $g_1 h_1 = h_1 g_1 \in G \times H$, then this commutes. But it is not always the case. Consider the fundamental group of figure eight generated by a, b, representing the left and right loop, $[ab] \neq ba$ in π_1(figure 8).

Definition 0.16 (Free products): Let $\{G_\alpha\}_{\alpha \in I}$ be a list of groups. A word in G_α is a finite sequence of length $m \geq 0$ of elements in G_α. If $m = 0$, we call the word the empty word.
We further define multiplication by concatenation.

$$(g_1, g_2) \cdot (g_3, g_3, g_4) = (g_1, g_2, g_3, g_3, g_4)$$

Note that the empty word is the identity element.

To make this a group, note that if g_i, g_{i+1} are in the same group and 1_α is the identity in G_α then we define elementary reduction tobe:

$$(g_1, \dots, g_i, g_{i+1}, \dots g_m) \iff (g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2} \dots g_m)$$

$$(g_1, \dots, g_{i-1}, 1, g_{i+1}, \dots g_m) \iff (g_1, \dots, g_{i-1}, g_{i+1}, g_{i+2} \dots g_m)$$

Two words are equivalent if they are related by the elementary reductions.
Now $(g_1, g_2, \dots, g_m)^{-1} = (g_m^{-1}, g_{m-1}^{-1}, \dots g_1^{-1}))$

Definition 0.17: Let $\{G_\alpha\}_{\alpha \in I}$ be a collection of groups. Then the free product, $*_{\alpha \in I} G_\alpha$ is the set of equivalence classes of words in G_α under the operation of concatenation. Note that we usually drop the parentheses and write $G * H$ for the product of two groups.

Example 4 Take two copies of $\mathbb{Z}_2$ generated by a and b respectively. Then $\mathbb{Z}_2 * \mathbb{Z}_2$ consists of elements that look like :

$$ababa\dots, \text{ or } bababa$$

because any two consecutive letters just cancel out. □

Definition 0.18 (Free group of rank n): An important class of free products are free groups of rank n, called F_n. They are formed by taking

$$\underbrace{\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \dots * \mathbb{Z}}_{n \text{ times}}$$

Let S be a set. If $\alpha \in S$, we can form an infinite cyclic group $\langle \alpha \rangle$. Let $\langle S \rangle = *_{\alpha \in S} \langle \alpha \rangle$. We call this the group generated by S.

Theorem 0.14 (Characteristic property of free groups):
Let S be a set. For any group H and any map

$$\phi : S \to H$$

there exist a unique homomorphism $\tilde{\phi} : \langle S \rangle \to H$ extending ϕ.

$$\begin{tikzcd} \langle S \rangle \arrow[dr, dashed, "\tilde{\varphi}"] & \\ S \arrow[u, hook] \arrow[r, hook, "\varphi"] & H \end{tikzcd}$$

Proof: Let G be a finitely generated group and let $\{g_1, \dots g_n\}$ be a generating set. Then the set map $\{g_1, \dots, g_n\} \hookrightarrow G$ given by inclusion extends to a map $\tilde{\phi} : \langle \{g_1, \dots, g_n\} \rangle \to G$. This map is surjective since $g_1, \dots g_n$ is a generating set. By the first isomorphism theorem, $F_n/R \simeq G$ where $R = \ker \tilde{\phi}$. $\square$

Definition 0.19: Let S be a set and let R be a set of words in S. Then the group given by presentation $\langle S \mid R \rangle$ is the group $\langle S \rangle / N$ where N is the normal closure of R in S (Smallest normal subgroup containing R). Sometimes we write $x = y$ for $xy^{-1} \in R$ and sometimes we write $x = 1$ for $x \in R$.

Example 5
- $\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = 1 \rangle$
- $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$
- $\mathbb{Z} * \mathbb{Z} = \langle a, b \mid \rangle$
- $D_n = \langle r, s \mid r^n = 1, s^2, sr = r^{n-1}s \rangle$ $\square$

Proposition 0.15:
If $G = \langle g_1, \dots, g_n \mid R_G \rangle$ and $H = \langle h_1, \dots, h_m \mid R_H \rangle$ then $G * H = \langle g_1, \dots, g_n, h_1, \dots, h_m \mid R_G, R_H \rangle$.

Example 6 $D_4 * \mathbb{Z}/2\mathbb{Z} = \langle r, d, a \mid r^4 = 1, s^2 = 1, sr = r^3 s, a^2 = 1 \rangle$. $\square$

Want: if $X = A \cup B$ then there is a map $\pi_1(A) * \pi_1(B) \to \pi_1(X)$ which is surjective.

Proposition 0.16:
If $\phi_\alpha : G_\alpha \to H$ is a collection of homomorphisms, then there exists a unique extension $\phi : *_\alpha G_\alpha \to H$.

Proof: The idea is $\phi(g_1, \dots, g_m)$ with $g_i \in G_i$, we have

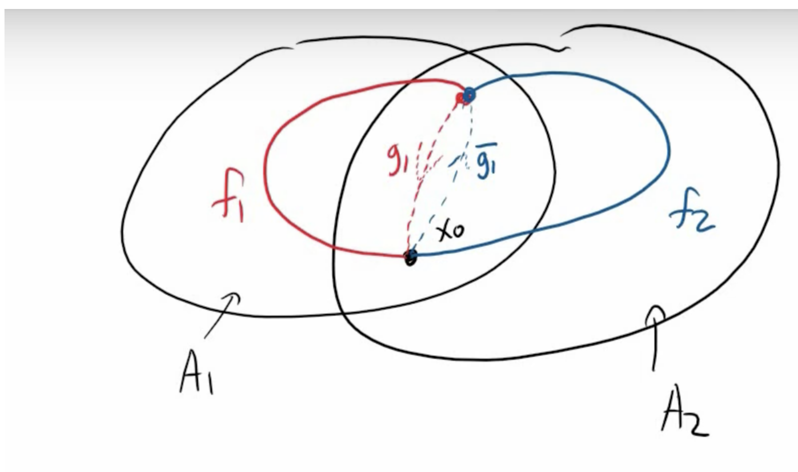
$$\phi(g_1, \dots, g_m) = \phi(g_1)\phi(g_2) \dots \phi(g_m) = \phi_1(g_1)\phi_2(g_2) \dots \phi_m(g_m)$$

□

Given $X = \bigcup_{\alpha \in J} A_\alpha$ with $x_0 \in A_\alpha$ for every α. The inclusion maps $i_\alpha : A \to X$ induce maps $i_{\alpha *} : \pi_1(A) \to \pi_1(X)$. So we get a map $I : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$.

Lemma 0.17:
Let $X = \bigcup_{\alpha \in J} A_\alpha$ where A_α is path connected, open and contains the basepoint x_0 for all α. Then every loop in X based at x_0 is homotopic to a loop which is a product of loops each of which is contained in a single A_α.

Proof: Let $f : I \to X$ be a loop based at x_0. Since it f is continuous and A_α is open for every α, $\bigcup_\alpha f^{-1}(A_\alpha)$ is a union of open intervals covering I.
Now I is compact, so every open cover of I has a finite subcover. This means we can split $[0, 1]$ into intervals $[0, s_1] \cup [s_1, s_2] \cup \ldots \cup [s_{m-1}, s_m]$. So that $f([s_{i-1}, s_i]) := f_i \subseteq A_i$.
So, $f \simeq f_1 \cdot \ldots \cdot f_m$. Next step is to make these into loops.



For example, in the image above, $f_1 \cdot f_2$ can be written as $(f_1 \cdot g_1) \cdot (\overline{g_1} \cdot f_2)$, where $(f_1 \cdot g_1)$ is a loop in A_1 and that $(\overline{g_1} \cdot f_2)$ is a loop in A_2.
Since A_i is path connected for all i, there is a path from $f(s_i)$ to the basepoint x_0. Call this path g_i.
Recall $g_i\overline{g_i}$ is homotopic to identity, so

$$f \simeq f_1 \cdot f_2 \cdot \ldots \cdot f_m \simeq \underbrace{(f_1 \cdot g_1)}_{\text{loop in } A_1} \underbrace{(\overline{g_1} \cdot f_2 \cdot g_2)}_{\text{loop in } A_2} (\overline{g_2} \cdot f_3 \cdot g_3) \ldots (\overline{g_{m-1} f_m})$$

So every loop f can be expressed as $a_1 \cdot a_2 \ldots a_m$ with $a_i \in A_i$. We have each $i(a_i) \in \pi_1(X)$, where i is the map induced by inclusion. □

check the following corollary and the theorem in textbook! important

Corollary 3: The map $I : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective under the conditions above.
By first isomorphism theorem, $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/\ker(I)$. Now, we want to understand, what is in the kernel of I? Suppose $w \in A_\alpha \cap A_\beta$. The maps

$$i_{\alpha\beta *} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$$

and

$$i_{\beta\alpha *} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\beta)$$

carry w into different groups and so different places in $*\pi_1(A_\alpha)$. By slight abuse of notation, call $i_{\alpha\beta}(w)$ the element which is the image of $i_{\beta\alpha *}(i_{\alpha\beta}(w)) \in \pi_1(X)$. $i_{\alpha\beta}(w) = i_{\beta\alpha}(w)$ in $\pi_1(X)$. So we need the relation $i_{\alpha\beta}(w) \cdot i_{\beta\alpha}(w)^{-1} = 1$.

The basic idea is that if you have something in $A_\alpha \cap A_\beta$, then if you include it in A_α first, then X or A_β first, then X, it will yield the same results. This is what is in the kernel.

Theorem 0.18 (Van Kampen's theorem):
Suppose X is a union of path connected open spaces A_α so that $X = \bigcup_{\alpha \in J} A_\alpha$. Suppose that for all $\alpha, \beta, \gamma \in J$, $A_\alpha \cap A_\beta$ and $A_\alpha \cap A_\beta \cap A_\gamma$ are path connected. Then the map

$$I : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective and the kernel of I is normally generated by elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)$ for $w \in \pi_1(A_\alpha \cap A_\beta)$. So $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N$ where N is the normal closure of elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)$.

Corollary 4: Let $X = A \cup B$ with A and B path connected. $A \cap B$ are path connected, and A and B are open.
Let $\pi_1(A) = \langle a_1, \ldots, a_n \mid R \rangle$ and $\pi_1(B) = \langle b_1, \ldots, b_m \mid S \rangle$ and $\pi_1(A \cap B) = \langle c_1, \ldots, c_\ell \mid T \rangle$.
Then,

$$\pi_1(X) = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \mid R, S, i_{AB}(c_1)i_{BA}(c_1^{-1}), \ldots i_{AB}(c_\ell)i_{BA}(c_\ell)^{-1} \rangle$$

Theorem 0.19:
$\pi_1(S^n) = 0$ for $n \geq 2$.

Proof: Let $S^n = \{(x_1, \ldots, x_{n+1} \subset \mathbb{R}^{n+1}) \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$.
Let $A = \{(x_1, \ldots, x_{n+1}) \subset S^n \mid x_{n+1} > 0.4\}$, and $B = \{(x_1, \ldots, x_{n+1}) \subset S^n \mid x_{n+1} < 0.6\}$.
Then, $A \cong D^n$ and $B \cong D^n$ with A and B are path connected and $A \cap B$ is path connected, and A and B are open.
Note that $A \cap B \cong S^{n-1} \times I$. By Van Kampen's theorem, since $\pi(A) = \langle e \rangle = \pi_1(B)$, then

$$\pi_1(S^n) = *_{i=1}^2 \{e\}/N = \{e\}.$$

Review textbook for this chapter
Review textbook for this chapter

Review textbook for this chapter
Review textbook for this chapter □

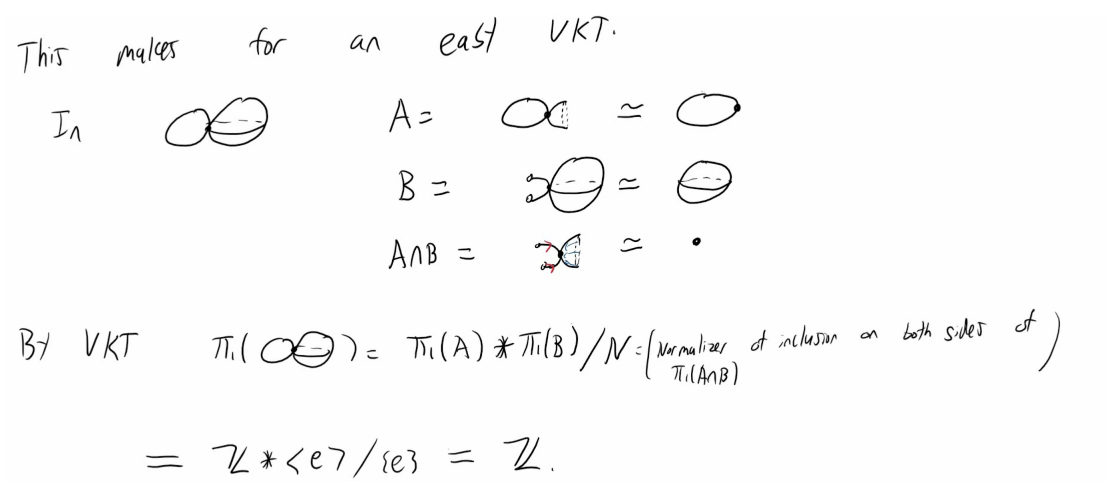
Lecture 8. Applications of Van Kampen's Theorem.

Wedge sums

Definition 0.20 (Wedge sum): Let $(X_1, x_1), \dots (X_n, x_n)$ be a collection of pointed spaces. Then the **wedge sum**, denoted $\bigvee_{i=1}^n x_i$ is the space

$$\coprod_{i=1}^{n} X_i / x_1 \sim x_2 \sim \dots \sim x_n$$

In many practical cases, points have a neighbourhood which deformation retracts onto them. (For examples, manifolds, finite CW complexes). This makes it easy to apply VKT.



Proposition:
Let $(X_1, x_1), \dots (X_n, x_n)$ be a collection of pointed spaces. Suppose each x_i has a neighbourhood which deformation retracts onto x_i, then

$$\pi_1(\bigvee_{i=1}^{n} X_i) = *_{i=1}^{n} \pi_1(X_i)$$

Proof: Let $x_i \in U_i \subset X_i$ be the neighbourhood of x_0 which retracts onto x_i. Let $A_i = X_i \bigvee (\bigvee_{j \neq i} U_j)$. Note that A_i is open in $\bigvee_{i=1}^{n} X_i$.
$A_i \cap A_j = U_i \cap U_j \simeq \{pt\}$ is path connected and has no π_1.
By VKT,

$$\pi_1(\bigvee_{i=1}^{n} X_i) = \pi_1(A_1) * \pi_1(A_2) * \ldots * \pi_1(A_n)/N$$

note that the N is trivial.
Since $A_i \simeq X_i$, we get $\pi_1(\bigvee +i = 1^n X_i) = \pi_1(X_1) * \pi_1(X_2) * \ldots * \pi_1(x_n)$.

<u>Cor</u> $\pi_1($ n circles $) \simeq F_n$.

□

Fundamental group of CW complexes

<u>Step 1</u> We will show that attaching an n cell for $n \geq 3$ does nothing to the fundamental group.

Proposition:
Let X be a path connected space and let $S^{n-1} \subset D^n$ for $n \geq 3$. Let $f : S^{n-1} \to X$ be a map and let $X' = X \coprod D^n / f(x) \sim x$. Then $\pi_1(X') = \pi_1(X)$.

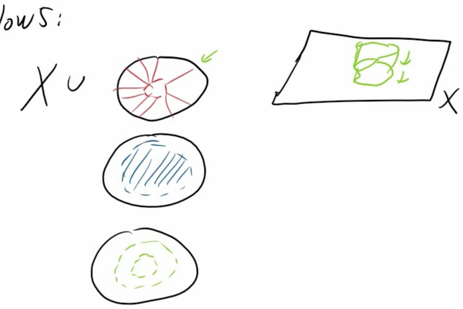
Proof:

<u>Proof</u> Take $A, B \subset X$ as follows:

$A = X \cup \{ y \in D^n \mid |y| > 1/3 \}$

$B = \{ y \in D^n \mid |y| < 2/3 \}$

$A \cap B$ is path connected.

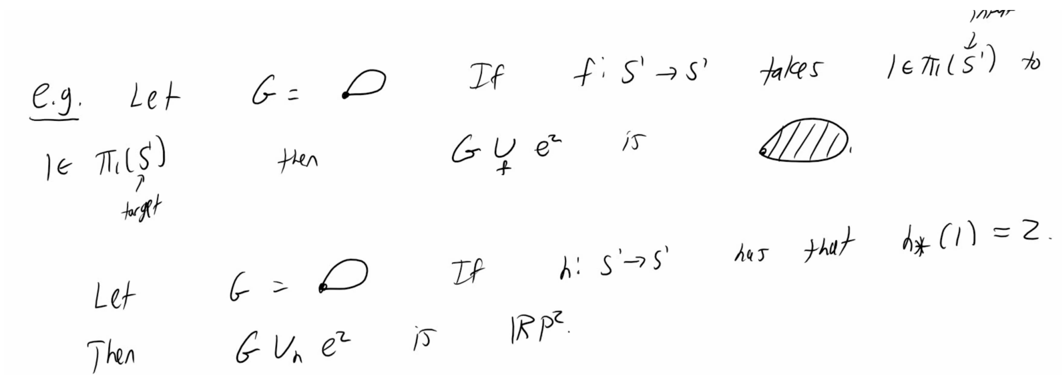


$A \simeq X, B \simeq D^n$, and $A \cap B \simeq S^{n-1}$.
So $\pi_1(X^1) = \pi_1(A) * \pi_1(B)/N$. Since S^{n-1} is simply connected for $n \geq 3$, we have $\pi_1(X') = \pi_1(X)$. Revisit this, i am not entirely sure.

□

<u>Step 2</u> So the interesting parts all happen when we attach 2 cells or dimensions lower. What happens when we attach 2 cells?
<u>Fact 1:</u>
$\pi_1(X') = F_n$ where x^1 is the 1 skeleton of a CW complex.

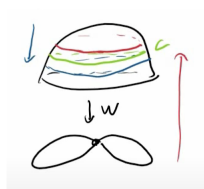
Fact 2:
If $x_f = x \cup_f y$ and $x_g = X \cup_g Y$, then if f and g are homotopic maps, $X_d \simeq X_g$. ()homotopy equivalent. Since a 2 cell is obtained by attaching $D^2 \to X^1$ by a map (up to homotopy) on S^1, attaching map is determined by an element of $\pi_1(x^n) = F_n$.



Proposition:
Let G be a connected graph and let e^2 be a 2 cell. Let $\pi_1(G) = \langle a_1, a_2, \ldots, a_n \mid \rangle$. And let $f : S^1 \to G$ be so that $f_* : \pi_1(S^1) \to \pi_1(G)$, $f_*(1) = w$. Then,

$$\pi_1(G \cup_f e^2) = \langle a_1, a_2, \ldots, a_n \mid w \rangle$$

Proof: Let $e^2 = \{y \in \mathbb{R}^2 \mid |y| \leq 1\}$. Let $A = G \cup \{y \in D^2 \mid y > 1/2\}$, let $B = \{y \in D^2 \mid y < 2/3\}$. As before, $A \simeq G, B \simeq D^2$, and $A \cap B \simeq S^1 \times I \simeq S^1$.



Want to know $i_{AB}(C)$ and $i_{BA}(C)$, where the former is mapping to A first and the latter is mapping to B first. We know $i_{BA}(c) = id$ since it shrinks to the top. We know $i_{AB}(c) = w$. So, $\pi_1(G \cup_f e^2) = \langle a_1, a_2, \ldots a_n \mid w = 1 \rangle$.

□

Corollary 5: Let X be a connected $2d$ cw complex. Suppose that $\pi_1(X^1) = F_n$. Suppose X is built with k 2-cells attached along $w_1, \ldots w_k$, then $\pi_1(X) = \langle g_1, \ldots, g_n \mid w_1, \ldots w_k \rangle$.

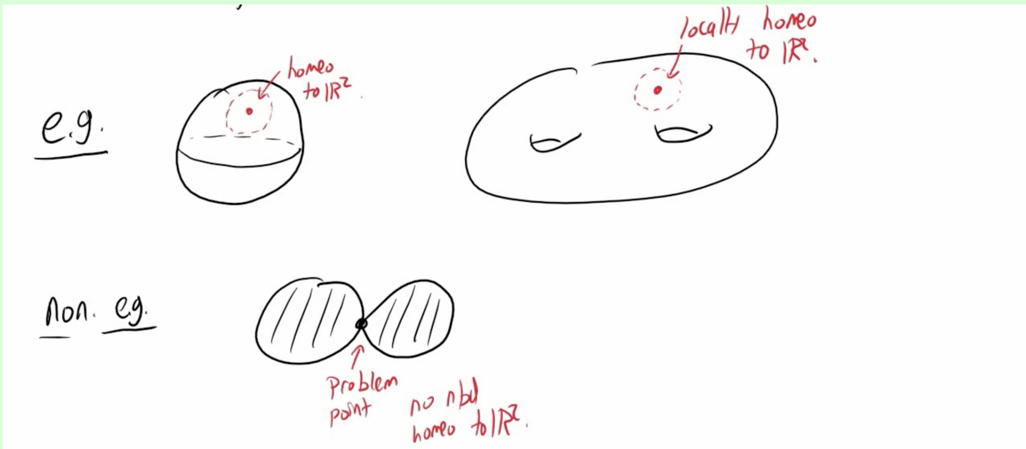
Definition 0.21: A group is finitely presented if it has a presentation with finite number of generators and relations.

Corollary 6: Every finitely presented group is $\pi_1(X)$ for some 2d CW complex X.

Lecture 9

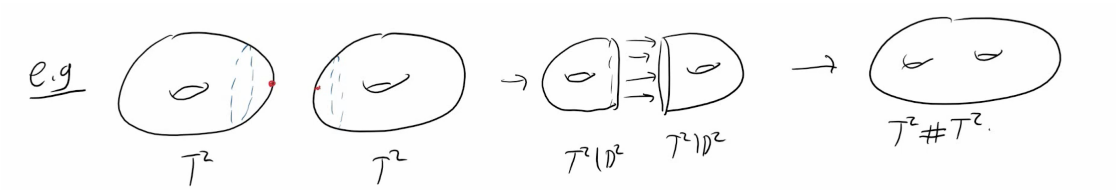
Surfaces

Definition: Surfaces are a special class of $2d$ CW complexes. They are finite CW complexes which are $2d$ and are locally homeomorphic to $\mathbb{R}^2$. For example



Surfaces are a examples of manifolds.

There is an important operation on surfaces, called the <u>connected sum</u> demoted by #.

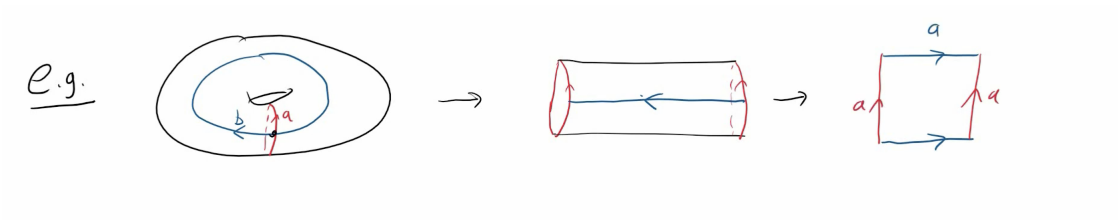


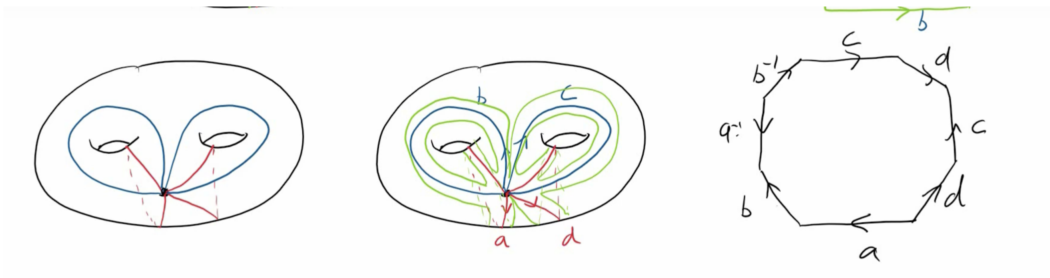
$A\#B$ is defined to be the surface obtained by removing $\mathbb{R}^2$ neighbourhoods of A and B surfaces, and gluing the resulting pieces together.
Question: does it matter the location of you taking out the D^2??

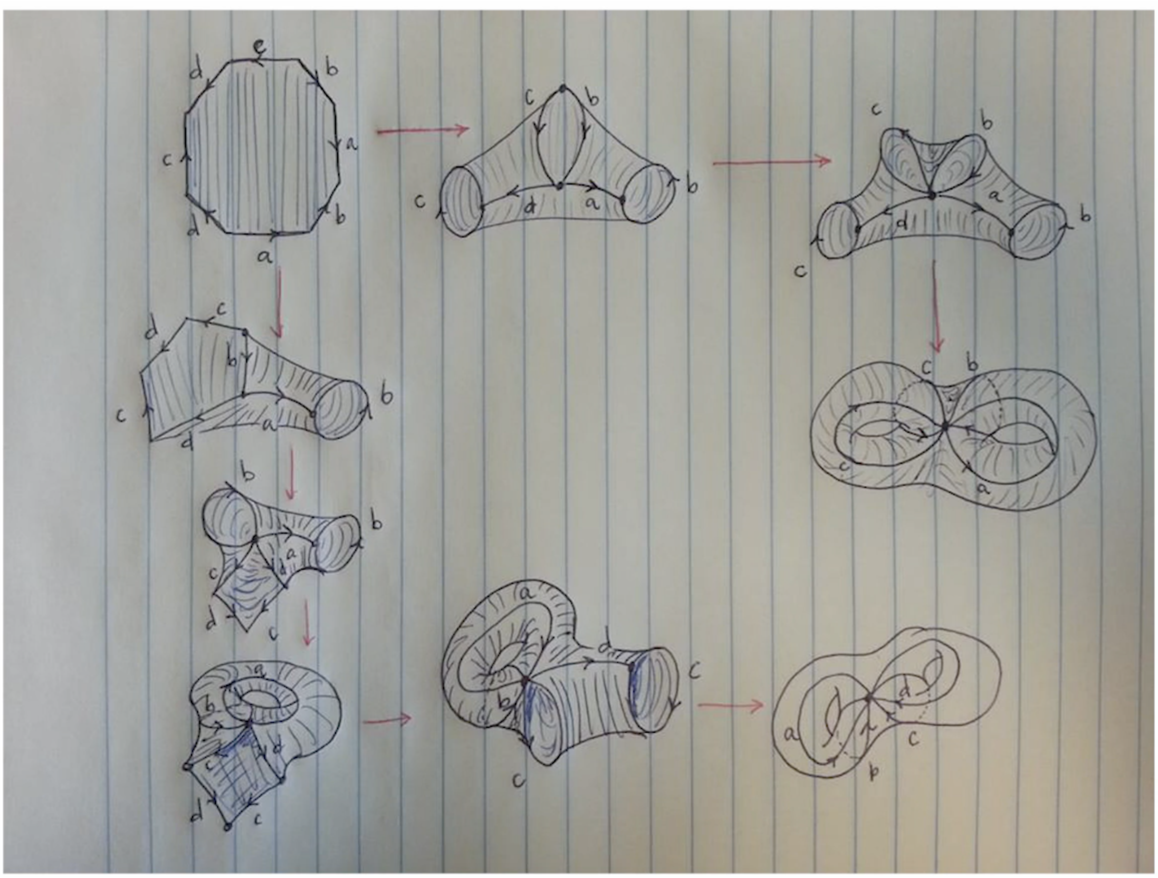
Note that like the above S^2 is an identity for #, as cutting out S^2 and gluing it back onn does nothing to the surface.
Hence S^2 is an identity for the operation of #.

Definition (The closed orientable surface of genus g): $\#^g T^2 = \underbrace{T^2 \# T^2 \# \ldots \# T^2}_{g \text{ times}}$ is called the closed orientable surface of genus g.

These spaces can be represented by polygons with gluing information.



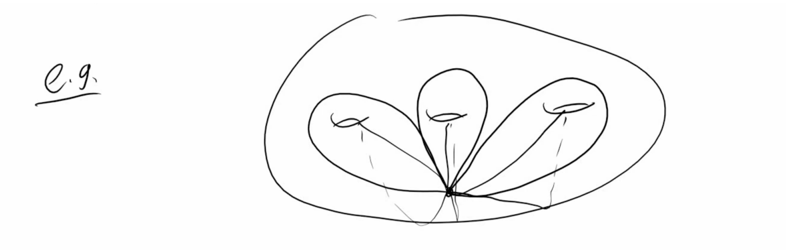




By corollary of VKT for 2 complexes, we have generators for every 1 cell. (the a, b, c, ds in the CW complex. We also have a relation for every 2 cell.) Note that $T^2 \# T^2$ is CW complex with one 0 cell, four 1 cell, and one 2 cell.
Recall that $[a, b] = aba^{-1}b^{-1}$. Where have we shown this at all?
So, if we denote the surface of $T^2 \# T^2$ by Σ_2, then we have

$$\pi_1(\Sigma_2) = \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle$$

Note that the above? why is the product of them equal to 1?attention. Should consult to Hatcher for sure!!
In general, we can do this for a genus g surface. In general the surface will be a $4g$ gon.



Denote $\Sigma_g = \#^g T^2$.

Theorem:

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g b_g] = 1 \rangle$$

Note that the $[a_1, b_1][a_2, b_2] \dots [a_g b_g] = 1$ expands into $aba^{-1}b^{-1} \dots$ is exactly traversing around the the genus g and getting back to the original point. That is why this long path formed this way is also in the genus.

Theorem (HARD theorem):
The proof of this hard theorem could take up to an entire course!
Every closed (no boundary), orientable, surface is of the form

$$\#^g T^2$$

for some g.

Corollary 7: Closed orientable surfaces are classified by their fundamental groups.

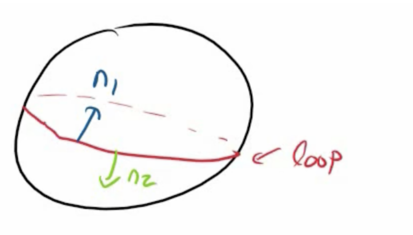
Proof:
$\Longrightarrow$:
That is, $\Sigma_g \cong_{\text{homeo}} \Sigma_h$, then by homeomorphic euqivalence invariance of π, we have that $\pi_1(\Sigma_g) = \pi_1(\Sigma_h)$.
$\Longleftarrow$:
Suppose $\pi_1(\Sigma_g) = \pi_1(\Sigma_h)$. Then we abelianize both fundamental groups.

$$Ab(\pi_1(\Sigma_g)) = Ab(\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle)$$

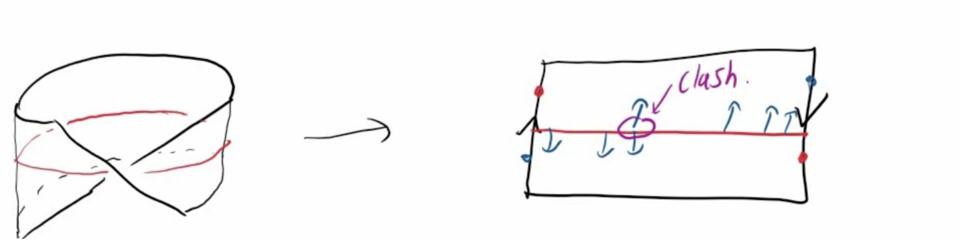
In the abelianization, $[a, b] = aba^{-1}b^{-1} = 1$. So the relation above is redundant. So $Ab(\pi_1(\Sigma_g)) = \mathbb{Z}^{2g}$
Similarly, $Ab(\pi_1(\Sigma_h)) = \mathbb{Z}^{2h}$. SInce $\mathbb{Z}^{2g} \cong_{is} \mathbb{Z}^{2h} \iff g = h$, $\Sigma_g = \Sigma_h$. □

0.3 Non-Orientable surfaces

In a surface, one can make two choices for the normal vector to a loop.



The question is: Can this always be done consistently? Does dragging the vector around the loop change its vector? The answer is no. Consider the mobius band.



A loop like this is called an orientation reversing loop.

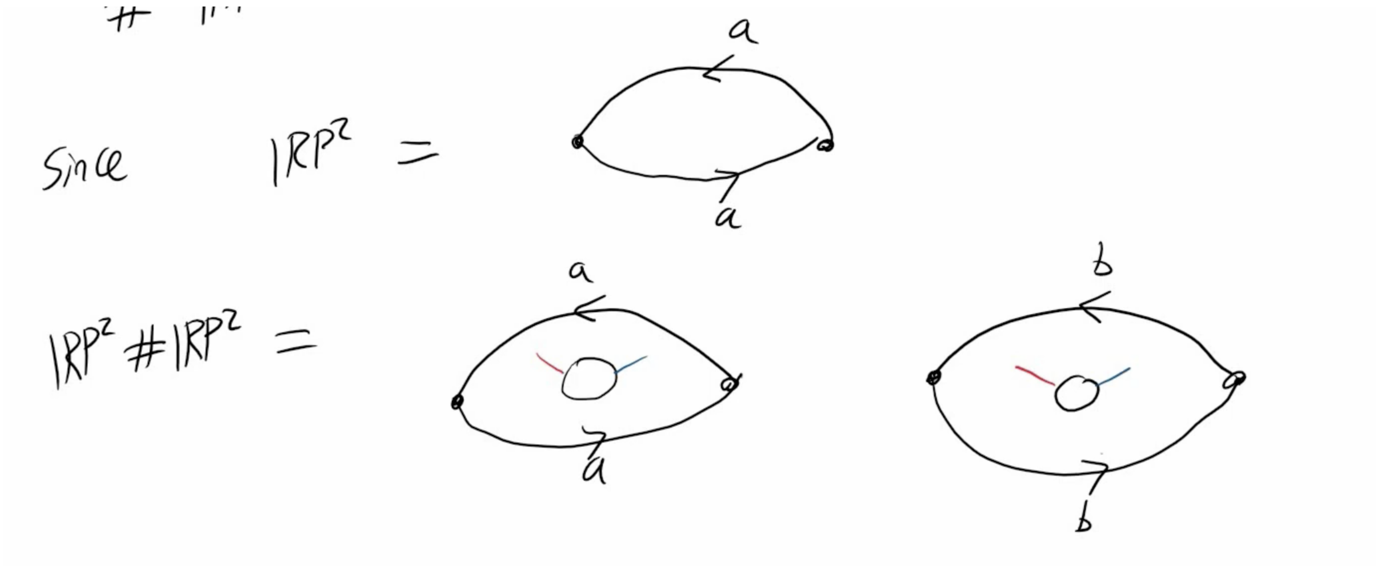
Definition (Orientable surface): A surface is orientable if it does not contain an orientation reversing loop. If a surface contains an orientable reversing loop, it is called non-orientable.

$\underline{\mathbb{R}P^2}$
Recall that $\mathbb{R}P^2 \cong S^2/x \sim -x$.

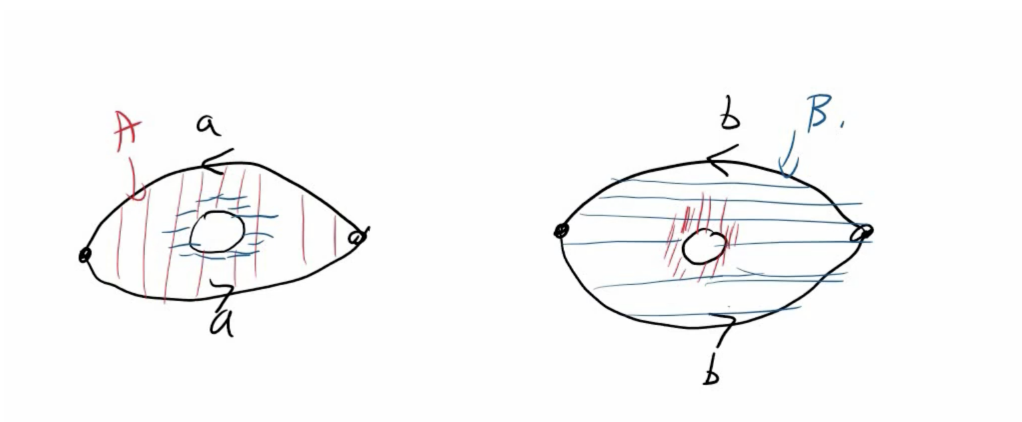
Note that the loop goes over a essentially twice. and the two antipodal points are now the same.
So $\mathbb{R}P^2 = e^0 \cup_g e^1 \cup_f e^2$ where g is the only possible mapping and g sends $1 \in \pi_1(S^1) \to a^2$.
By previous corollary to VKT for 2 complexes, we get

Proposition 0.20:
$\pi_1(\mathbb{R}P^2) = \langle a \mid a^2 = 1\rangle = \mathbb{Z}/2\mathbb{Z}$.

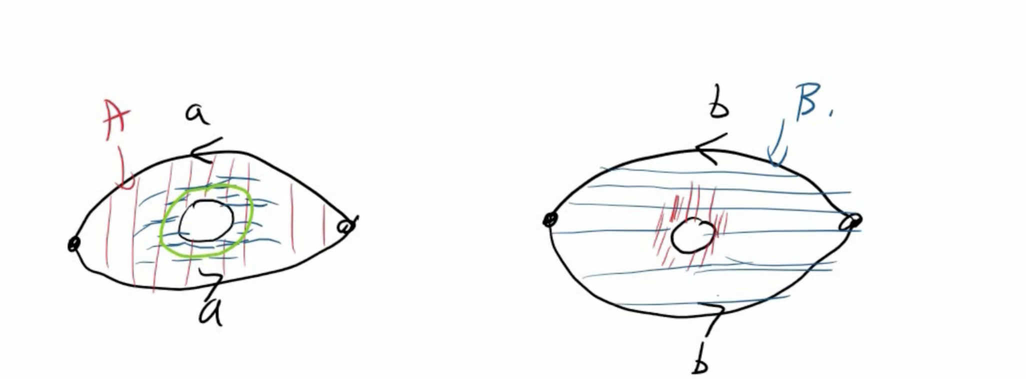
Theorem 0.21 (Hard theorem again):
Every connected compact closed non-orientable surfaces is homeomorhic to $\#^k \mathbb{R}P^2$ for some $k \in \mathbb{N}$.



The blue and red lines represents the "sticking". That is, if you go in the hole of red line, then you pop out from the red line on the right.
To apply the VKT, let A be the entire surface in the left while it pops out a little bit on the set on right. Similarly right theo ther ones.



Now imagine what $A \cap B$ looks like. Note that $A \cap B = S^1 \times I$ (consider the green circle. It expands out to cover the blue/red parts and it shrinks and goes to the other set to cover the blue/ red points.)



Now, $A = S^1 \times I \simeq S^1$, $B = S^1 \times I \simeq S^1$.
So all of $A \cap B, A, B$ have fundamental group of $\mathbb{Z}$. So

$$\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2) = \langle a, b \mid a^2 = b^2 \rangle$$

Usually, we sub in a_1 for a, to and get a_2^{-1} for b, to get

$$\langle a_1, a_2 \mid a_1^2 a_2^2 = 1 \rangle$$

Proposition:
$\pi_1(\#^k \mathbb{R}P^2) = \langle a_1, \ldots, a_k \mid a_1^2 a_2^2 \ldots a_k^2 = 1 \rangle$.

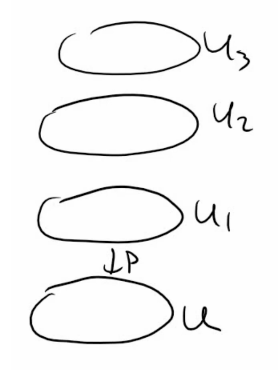
Fact: $Ab(\pi_1(\#^k \mathbb{R}P^2)) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z} \neq \mathbb{Z}^{2g}$.

Corollary 8:
This is a very cool corollary!
Closed, connected, compact surfaces are classified by their fundamental groups.

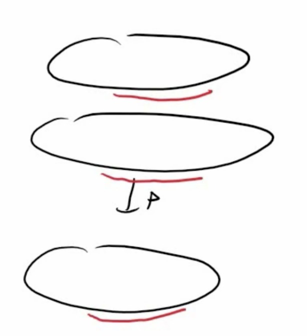
Lecture 10. More about covering space.

Recall definition of a covering space.

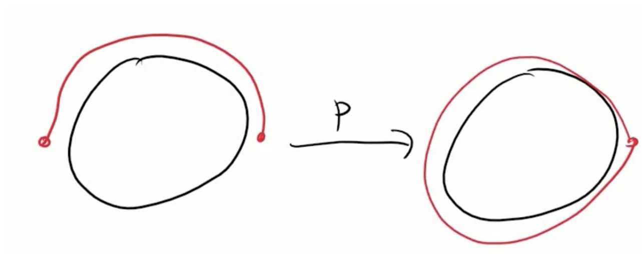
Definition: Let X be a topological space. A covering space is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that $\forall x \in X$, there is a neighbourhood $U \ni x$ such that $p^{-1}(U) = \coprod_{\alpha \in I} U_\alpha$, a disjoint collection of sets U_α mapping so that p maps each set in $\coprod_{\alpha \in I} U_\alpha$ homeomorphically onto U.



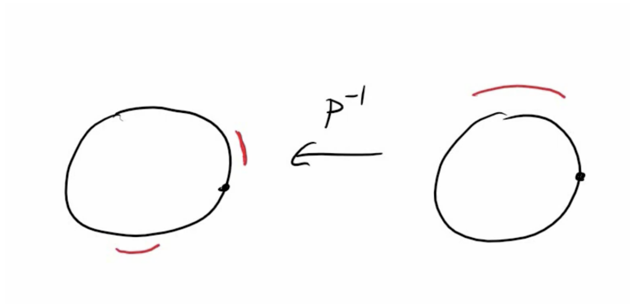
Each U in this set up is called an evenly covered neighborhood and each U_i is called a sheet. If $p^{-1}(U)$ has n sheets, it is called an n fold cover.
(boring cover) 2 fold cover of $S^1 : p(S^1 \coprod S^1) \to S^1$ by



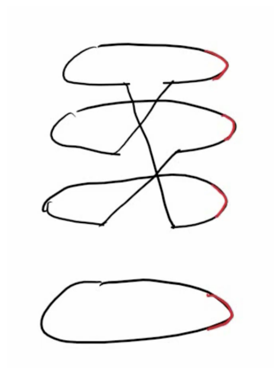
Disconnected covers are just disjoint unions of connected covers, which is really boring. So we just look at connected covers.
(More interesting cover)
The 2 fold cover $S^1 \to S^1$. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Define $p : S^1 \to S^1$ by $p(z) = z^2$ $(\theta \to 2\theta.)$



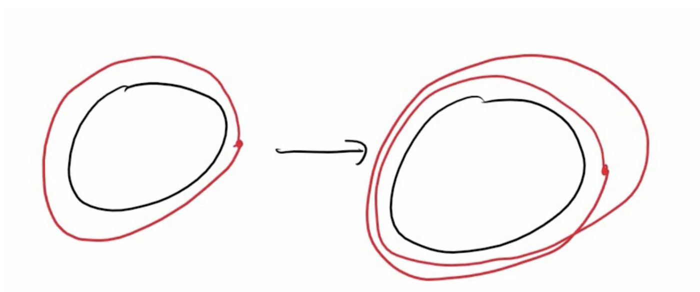
We are particularly interested in the inverse image of p.



Consider the n fold cover $S^1 \to S^1$, $p_n : S^1 \to S^1, z \mapsto z^n$. Below is an alternative visualization of the connected cover of the n circles.



Example: $p_* : \pi_1(\tilde{X}.\tilde{x_0}) \to \pi_1(X, x_0)$ for $p_2 : S^1 \to S^1, z \mapsto z^2$, is the induced homomorphism. Note that the image of p_{2*} is equal to $2\mathbb{Z}$. Similarly, p_{3*} has the image $3\mathbb{Z}$.



Now, recall the homotopy lifting property. If $p : \tilde{X} \to X$ is a cover and $f_t : Y \to X$ is a homotopy, and suppose that $\tilde{f}_0 : Y \to \tilde{X}$ is a lift of f_0. Then there is a unique lift $\tilde{f}_t : Y \to \tilde{X}$ when looking at a loop in X. The loop can lift to either a path or a loop.
For example, the path that walks the half circle in RHS is a loop when it is lifted to the space to the left.

Injectivity of induced maps

Proposition:
The map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ induced by a covering map is injective. Moreover, the image subgroup

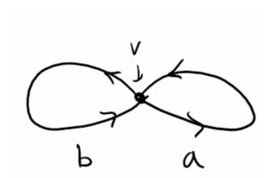
$$p_*(\pi_1(\tilde{X}, \tilde{x_0})) \leq \pi_1(X, x_0)$$

consists of loops in (X, x_0) which lifts to loops in $(\tilde{X}, \tilde{x_0})$.

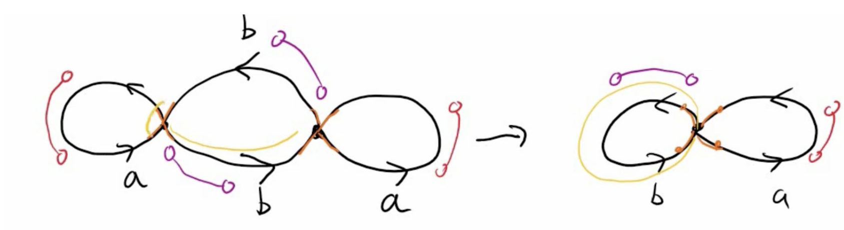
Proof: Suppose $\tilde{f_0} : I \to \tilde{X}$ is a loop so that $f_0 = p \circ \tilde{f_0}$ is trivial in $\pi_1(X, x_0)$. That is, it is in the kernel of p_*. Then there exists a homotopy f_t from f_0 to the trivial loop. By the homotopy lifting principle, we can lift this to a homotopy $\tilde{f_t}$ of $\tilde{f_0}$ since f_1 is constant so is $\tilde{f_1}$, so $\tilde{f_0}$ is homotopic to a constant loop.
So $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.
If a loop $f : I \to X$ lifts to a loop $\tilde{f} : I \to \tilde{X}$, then clearly $p_*(\tilde{f})$ is in the image of p. Conversely, if $\tilde{f} : I \to X$ is a loop, then $p(\tilde{f})$ lifts to the loop $\tilde{f}$ once we specify that $p^{-1}(x_0) = \tilde{x_0}$. $\square$

Coverings of $S^1 \bigvee S^1$.

Recall that $\pi_1(S^1 \vee S^1) = F_2$ with a basis given by $[a]$ and $[b]$.
At the vertex v a covering space must have a collection of disjoint 4 valent vertices. Moreover, the edges must project so that 2 edges go to a (one in and one out), and same for b (one in and one out.)



Note that the space on the LHS is a covering space for the one on the RHS.



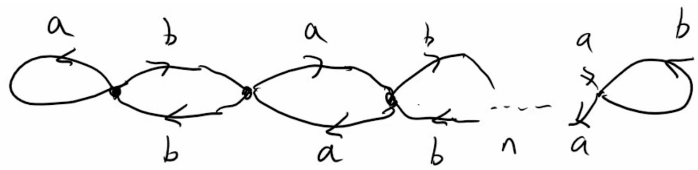
This is a covering map, as the yellow edge projects to the yellow, (b) and the a edges project to the a edges. The preimage of the red set is the two disjoint red sets on the RHS. The purple set's preimage is the two purple sets on the left. Moreover, the four valences edges for the vertex has the preimage of the two little cross in the left.
The LHS is homotopy equivalent to the wedge of 3 circles, whereas the RHS is the wedge of two circles. By the proposition above, the free group of 3 generators is a subgroup of the free group of 2 generators.

$\pi_1($  $, \cdot) = F_3 = \langle a, b^2, bab^{-1} \rangle$

But recall that $p_* : \pi_1($  $, \cdot) \to \pi_1($  $, \cdot)$ is injective. Then a, b^2, bab^{-1} generate a free group on 3 generators in the free group $\langle \{a, b\} \rangle$.

> **Corollary:** F_3 is a subgroup of F_2.

Note that this is actually related to some finite automata theory, where two generators are enough to represent all sorts of group structures! (Construct the free groups and then use the corresponding relations.) In fact, we can generalize the above and

$G_n =$ 

This is a cover of  as before

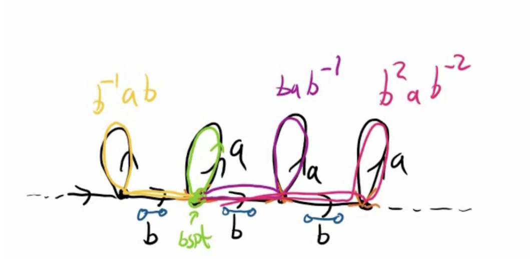
But $\pi_1(G_n) = F_n$

> **Corollary:** F_n is a subgroup of F_2. For all $n \in \mathbb{N}$.

Now, consider the following space

$G_\infty =$ 

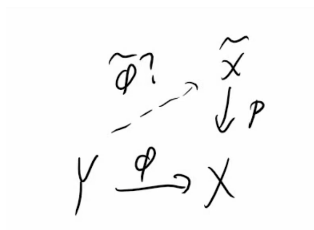
> **Corollary:** F_∞ is a subgroup of F_2.



The generators map to $\langle b^n a b^{-n} \mid n \in \mathbb{Z} \rangle$

More general theory

Here is a natural question: Given a map $\psi : Y \to X$. When does ψ lift to a map $\tilde{\psi} : Y \to \tilde{X}$ for a cover $p : \tilde{X} \to X$?



Proposition 0.22:
Let $p : \tilde{X} \to X$ be a covering map and let Y be a path connected and locally path connected space. Then a map

$$\psi : (Y, y_0) \to (X, x_0)$$

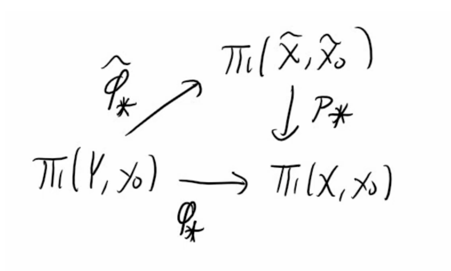
lifts to a map $\tilde{\psi} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if

$$\psi_*(\pi_1(Y, y_0)) \subset P_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

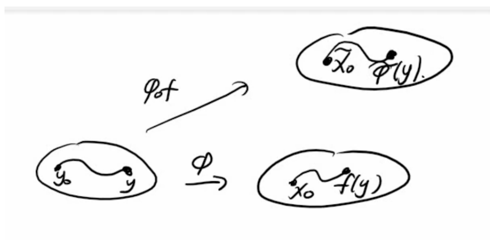
Note that elements from the left are in $\pi_1(X)$ so are the elements from the right are in $\pi_1(X)$. This inclusion makes sense.

Proof: $\Longrightarrow$
Assume there is a lift $\tilde{\psi}$. Then the following maps commute.

$$P_* \circ \tilde{\psi}_* = \psi_*$$



So ψ_* is in the image of P_*.
$\Longleftarrow$
Suppose that ψ_* is in the image of P_*.
Recall the path lifting property. For any path $f : I \to X$ with $f(0) = x_0$ and any choice of point $\tilde{x_0} \in p^{-1}(x_0)$, there exists a unique lift of f starting at $\tilde{x_0}$.
For every point $y \in Y$, choose a path f_y so that $f(0) = y_0$ and $f(1) = y$. This is possible since y is path connected. Then, $\psi \circ f_y$ is a path from $\psi \circ f(0) = \psi(y_0) = x_0$ to $\psi \circ f(1) = \psi(y)$.
Lift this path to a path $\psi \tilde{\circ} f_y$ starting at $\tilde{x_0}$.



Define $\tilde{\psi}(y) = \psi \circ f_y(1)$.
We should be worried because there are some technnicalities to address.
Claim 1. $\tilde{\psi}$ is well defined. (We started off with arbitrarily choice of path, we need to show that the path does not matter.)
Proof of claim 1. Suppose that f, f' are two paths from y_0 to y in Y. Then $f' \cdot \overline{f}$ is a loop based at y_0 in Y.
By our assumption,

$$\psi_*(f' \cdot \overline{f}) \in P_*(\pi_1(\tilde{X}, \tilde{x_0}))$$

So, $\psi(f' \cdot \overline{f})$ is homotopic to some loop of the form $p \cdot g$, where g is a loop based in $\tilde{x_0}$.
So, $p \circ g \sim \psi \circ (f' \cdot \overline{f}) = \psi(f') \circ \overline{\psi(f)}$.
Composing with $\psi \circ f$ on both sides, we get $(p \circ g) \cdot (\psi \circ f) \simeq \psi \circ f'$, as paths from x_0 to $f(y)$.
By uniqueness of path lifting, the lifts starting at $\tilde{x_0}$ end at the same point. The lift of $p \circ g$ is just g, a loop based at $\tilde{x_0}$. So

$$\widetilde{\psi \circ f'}(1) = g \circ \widetilde{\psi \circ f}(1) = \widetilde{\psi \circ f}(1)$$

so $\tilde{\psi}(y)$ is independent of the choice of path.
Claim 2 $\tilde{\psi}$ is continuous (using continuity of ψ and evenly covering neighbourhoods). How about uniqueness?

Proposition:
Given a covering space $p : \tilde{X} \to X$ and a map $\phi : Y \to X$. If two lifts $\tilde{\psi}_1$ and $\tilde{\psi}_2$ of ψ agree at a single point of y, and y is connected, then $\tilde{\psi}_1$ and $\tilde{\psi}_2$ agree on all of Y.

$\square$

Lecture 11

Recall that if $p : \tilde{X} \to X$ is a covering map, then $P_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.
A natural question is: Is every subgroup of $\pi_1(X, x_0)$ given by $P_* : \pi_1(\tilde{X}) \to \pi_1(X)$ for some cover $\tilde{X}$.
Start: Is there a simply connected space $\tilde{X}$ so that $\tilde{X}$ covers X. This is the hardest case and other groups will follow from it.
We need some conditions on X for it to have a simply connected covering space. In particular, if $U \subset X$ is an evenly covered neighbourhood and $\gamma \in U$ is a loop, then $p^{-1}(\gamma)$ is a loop in $p^{-1}(u)$ (for some simply connected covering $p : \tilde{X} \to X$.) Since $\tilde{X}$ is simply connected, $p^{-1}(\gamma)$ contracts to a point. Composing with p, we get a null homotopy of $\gamma \subset X$.

Definition 0.22: A topological space X is semilocally simply connected (slsc), if every point x has a neighbourhood u so that the inclusion map $i : U \to X$ induces the trivial map $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$.

For example, all CW complexes are slsc. (in fact they are locally contractible.
For example, the Hawaiian earring is the subspace of $\mathbb{R}^2$ given by

$$\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = (1/n)^2\}$$

For example, at the origin, no neighbourhood which includes into $\pi_1(X)$ trivially, because each neighbourhood has infinitely many generators. In addition, this is different than the wedge of ∞ circles $\bigvee_{i=1}^{\infty} S^1$. Are they even homeomorphic? why is it different than the circles?

Definition 0.23: A covering space $p : \tilde{X} \to X$ is called a universal cover if $\tilde{X}$ is simply connected.

Recall a space is locally path connected (lpc) if every point has a path connected neighbourhood.

Theorem 0.23:
Every path connected, lpc, slsc, topological space X, has a universal covering space.

Proof: Choose a basepoint $x_0 \in X$. Define $\tilde{X}$ (as a set) to be the set of homotopy classes of paths □

Lecture 11

Recall that if $p : \tilde{X} \to X$ is a covering map, then $P_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.
A natural question is: Is every subgroup of $\pi_1(X, x_0)$ given by $P_* : \pi_1(\tilde{X}) \to \pi_1(X)$ for some cover $\tilde{X}$.
Start: Is there a simply connected space $\tilde{X}$ so that $\tilde{X}$ covers X. This is the hardest case and other groups will follow from it.
We need some conditions on X for it to have a simply connected covering space. In particular, if $U \subset X$ is an evenly covered neighbourhood and $\gamma \in U$ is a loop, then $p^{-1}(\gamma)$ is a loop in $p^{-1}(U)$ (for some simply connected covering $p : \tilde{X} \to X$.) Since $\tilde{X}$ is simply connected, $p^{-1}(\gamma)$ contracts to a point. Composing with p, we get a null homotopy of $\gamma \subset X$.

Definition 0.24: A topological space X is semilocally simply connected (slsc), if every point x has a neighbourhood u so that the inclusion map $i : U \to X$ induces the trivial map $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$.

For example, all CW complexes are slsc. (in fact they are locally contractible.
For example, the Hawaiian earring is the subspace of $\mathbb{R}^2$ given by

$$\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = (1/n)^2\}$$

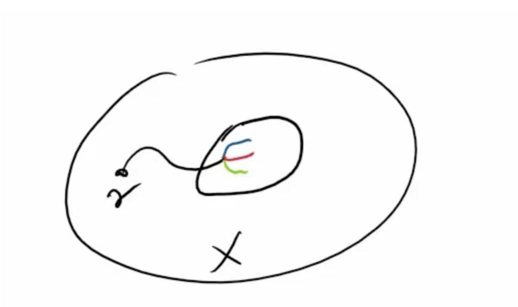
is not SLSC.
For example, at the origin, no neighbourhood which includes into $\pi_1(X)$ trivially, because each neighbourhood has infinitely many generators. In addition, this is different than the wedge of ∞ circles $\bigvee_{i=1}^{\infty} S^1$. Are they even homeomorphic? why is it different than the circles?

Definition 0.25: A covering space $p : \tilde{X} \to X$ is called a universal cover if $\tilde{X}$ is simply connected. One part why it is important, is that a universal cover space covers every other connected cover of X! So the universal cover space is worthy of studying.

A necessary condition for X to have a simply connected covering space is that it must be SLSC. So each point has a neighbourhood, such that the induced group from its neighbourhood into the induced group of X is trivial. Why must we satisfy this condition if we want a simply connected covering space? Because every $x \in X$ has a neighbourhood U which lifts to $\tilde{U} \subset \tilde{X}$ that is projected homeomorphically by p, and each loop in U lift to a loop in $\tilde{U}$. The lifted loop is nullhomotopic in $\tilde{X}$ so the original loop is nullhomotopic in X.
Recall a space is locally path connected (lpc) if every point has a path connected neighbourhood.

Theorem 0.24:
Every path connected, lpc, slsc, topological space X, has a universal covering space.

Proof: Choose a basepoint $x_0 \in X$. Define $\tilde{X}$ (as a set) to be the set of homotopy classes of paths starting at x_0. So $\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0\}$.
Topology on $\tilde{X}$
Now, we put a topology on $\tilde{X}$. Let U be a set in X so that $i_* : \pi_1(u) \to \pi_1(x)$ is trivial. Suppose that γ is a path which start at x_0 and ends in U. Let $U_{[\gamma]} = \{[\gamma \cdot \nu] \mid \nu \text{ is a path in } U \text{ which starts at } \gamma(1)\}$



That is, the place where it splits is $\gamma(1)$, and ν is a path in U starting at $\gamma(1)$, which is the three little colorful strands.
Check that this is a neighbourhood basis of $\tilde{X}$.
Now we define a map $p : \tilde{X} \to X$ so that $p([\gamma]) = \gamma(1)$. The previously defined topology makes p continuous.
$\tilde{X}$ is simply connected
Let $[\nu] \in \tilde{X}$ be a point. Let ν_t be a path in X given by

$$\nu_t = \begin{cases} \nu & \text{on } [0,t] \\ \nu(t) & \text{on } [t,1] \end{cases}$$

Then the map $f : I \to \tilde{X}$ given by $t \to [\gamma_t]$ is a path from $[\gamma_0] = [x_0]$ to $[\gamma_1] = [\gamma]$. Next we want that $\pi_1(\tilde{X}, [x_0]) = 0$. Since P_* is injective, we only need to show that $P_*(\pi_1(\tilde{X}))$ is trivial.
Recall that elements in the image of P_* are those loops in X which lifts to loops in $\tilde{X}$. A loop $\gamma \in X$ lifts to the loop $t \mapsto [\gamma_t]$. Since this is a loop in $\tilde{X}$, $[\gamma_1] = [\gamma_0] = [x_0]$ so $[\gamma_1] = [\gamma]$ is trivial. So $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = 0$ so $\pi_1(\tilde{X}, \tilde{x_0}) = \{e\}$.
$\tilde{X}$ is a cover Send the set $u_{[\gamma]}$ to u, then $p : U_{[\gamma]} \to U$ is a homeomorphism. While existence is nice, it doesn't tell us how to get $\tilde{X}$ in a useful form.
I should review this proof? □

Proof: Alan Hatcher's Proof
The motivation of this construction is the follows: suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a simply connected covering space. Each point $\widetilde{x} \in \widetilde{X}$ can be joined by $\tilde{x_0}$ by a unique homotopy of paths (since it is simply connected.) We view $\tilde{X}$ as a homotopy cllasses of paths starting at $\tilde{x_0}$. We describe $\tilde{X}$ in terms of X.
Given a pc, lpc, slsc space X with basepoint $x_0 \in X$, we define

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

$[\gamma]$ denotes the homotopy class of γ as a path, that is the homotopy of paths fixing the endpoints. We define the function

$$p : \widetilde{X} \to X, [\gamma] \mapsto \gamma(1)$$

Since it is a homotopy of paths, the endpoint is well defined, so mapping to $\gamma(1)$ is well defined. Since X is path connected, the endpoint $\gamma(1)$ can be any point of X, so p is surjective. In other words, all points in X is hit.
(Some proofs omitted). Consider $\mathcal{U}$, the collection of path connected open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. It turns out that $\mathcal{U}$ is a basis for the topology on X if X is locally path connected and semilocally simply connected.
Now given a set $U \in \mathcal{U}$, (that is one of the little neighbourhoods), and a path γ inn X, from x_0 to a point in U, let

$$U_{[\gamma]}\{[\gamma \cdot \nu] \mid \nu \text{ is a path in } U \text{ with } \nu(0) = \gamma(1)\}$$

These are the paths that are path homotopic to the paths that start at x_0, goes to $\gamma(1)$, then go to another point in U. Observe that $p : U_{[\gamma]} \to U$ is surjective.
It then follows that $U_{[\gamma]}$ is a basis for a topology on $\tilde{X}$, $p : U_{[\gamma]} \to U$ is a homeomorphism.
The rest of the proof follows from the previous proof provided by the prof. □

Proposition 0.25:
If $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ are covering spaces, then so is

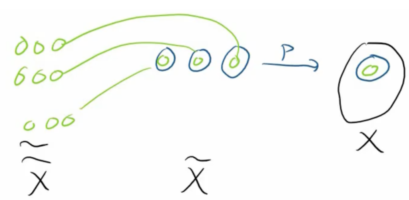
$$p \times q : \tilde{X} \times \tilde{Y} \to X \times Y, (x, y) \mapsto (p(x), q(y))$$

$$id \times q : \tilde{X} \times \tilde{Y} \to \tilde{X} \times Y, (x, y) \mapsto (x, q(y))$$

$$p \times id : \tilde{X} \times \tilde{Y} \to X \times \tilde{Y}, (x, y) \mapsto (p(x), y)$$

Proposition 0.26:
A composition of covering maps is a covering map.

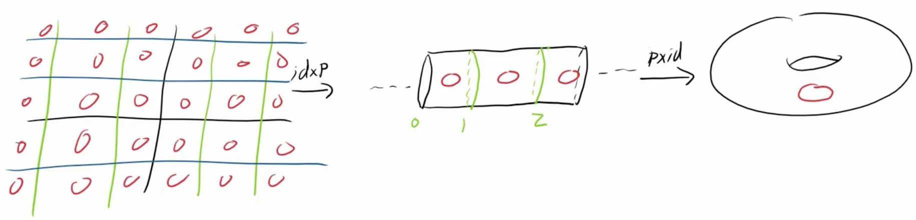
Idea.



Recall: $p: \mathbb{R} \to S^1$ is a covering map so since $\mathbb{R}$ is contractible $\mathbb{R}$ is the universal cover.

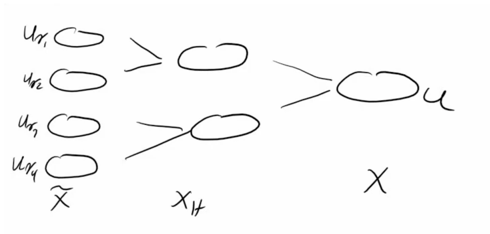
So $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a universal cover
$\mathbb{R}^2 \to T^2$.

also $p \times id : \mathbb{R} \times \mathbb{R} \to S^1 \times \mathbb{R}$ a u.c. of $S^1 \times \mathbb{R}$.



Proposition 0.27:
Suppose X is a path connected, l.p.c., slsc. Then for every $H \leq \pi_1(X, x_0)$ there exist a covering space $p_H : X_H \to X$ so that $P_{H*}(\pi_1(x_H, \tilde{x_0})) = H$ for some basepoint $\tilde{x_0} \in X_H$.

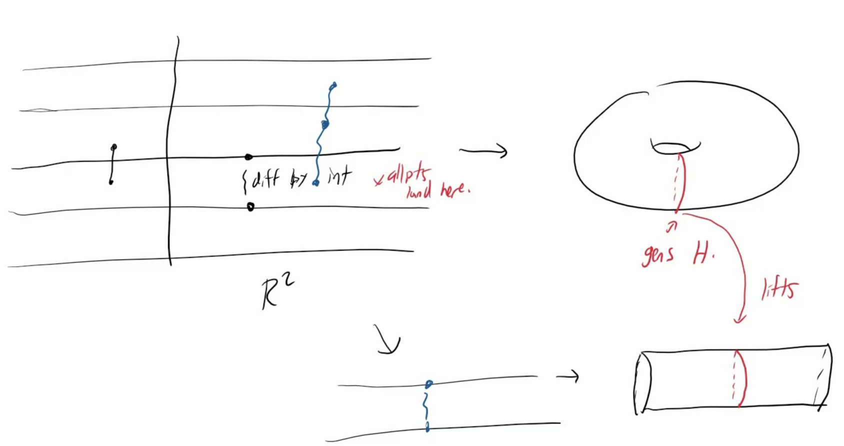
Proof: Let $\tilde{X}$ be the universal cover of X. Define an equivalence relation on points $[\gamma], [\gamma'] \in \tilde{X}$ by $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \overline{\gamma'}] \in H \leq \pi_1(X)$.
Check this is an equivalence relation (based on the fact that H is a subgroup.)
Let $X_H = \tilde{X}\,\tilde{}$. Let $u \subset X$ be a slsc evenly covered neighbourhood. Recall a neighborhood basis for $\tilde{X}$ near γ is given by $U_\gamma = \{[\gamma \cdot \nu] \mid \gamma(1) = \nu(0) \text{ and } \nu \subset U\}$.
Note that if $\nu(1) = \nu'(1)$ then ν is homotopic to ν' if and only if $\gamma \cdot \nu$ is homotopic to $\nu' \cdot n$. Then if $[\nu] = [\nu']$ then the entire nbd $U_{[\gamma]} = U_{[\gamma']}$.
So the map $p : \tilde{X} \to X$ given by $[\gamma] \mapsto \gamma(1)$ descends to a well defined covering map $P_H : X_H \to X$.



Recall the image of $\pi_1(X_H, \tilde{x_0})$ under P_H consists of loops in X which lifts to loops in X_H. Take $\tilde{x_0}$ to be the class of constant paths at x_0.
If $\gamma \in X$ lifts to a loop in X_H starting at $[x_0]$ and ending at $[\nu]$, then $[\nu] \sim [x_0]$, where x_0 is constant path, so $[\nu \cdot \vec{x}_0] \in H \implies [\nu] \in H$.

$\square$

Example: recall $p : \mathbb{R}^2 \to T^2$ is a universal cover if
$H = id \times \mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$ be the relation on $\mathbb{R}^2$.

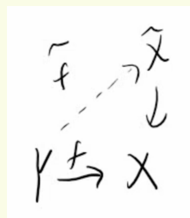


Lecture 12 Deck Transformations

Let's give a couple reminders about where we left off:

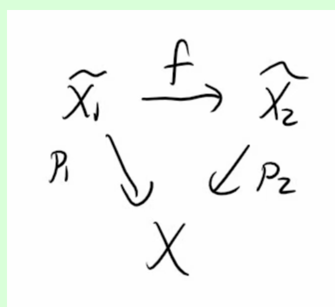
Proposition:
Give a path connected, (PC), locally path connected (LPC), semilocally simply connected (SLSC) space, X:

1. If $p : \widetilde{X} \to X$ is a covering space then $p_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ is injective.
2. For every subgroup $h \leq \pi_1(X)$, there exists a covering space $\widetilde{X_H}$, such that $p : \widetilde{X_H} \to X$ so that $p_*(\pi_1(\widetilde{X})) = H$.
3. Elements in the image of p_* are loops in X that lifts to loops in $\widetilde{X}$.
4. A map $f : Y \to X$ lifts to $\widetilde{X}$ if and only if $f_*(\pi_1(Y)) \subset P_*(\pi_1(\widetilde{X}))$.

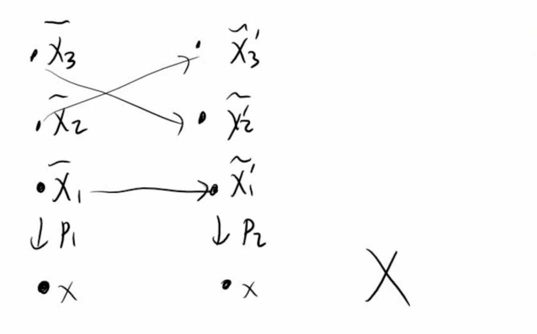


The equivalence relation on covering spaces is called a covering space isomorphism.

Definition 0.26: Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be covering spaces. Then a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ is a covering space isomorphism if $p_2 \circ f = p_1$.



This means $p_1^{-1}(x)$ is sent to $p_2^{-1}(x)$ for all $x \in X$. In this image below, the points that the maps are sent to would be a permutation.



Proposition:
If X is p.c. and l.p.c. then two p.c. covering spaces are isomorphism via an isomorphiism taking basepoints $\tilde{X_1}$ to $\tilde{X_2}$ if and only if $p_{1*}(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = p_{2*}(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$.
In other words, covering spaves are isomorphic if their induced maps land in the same subgroup.

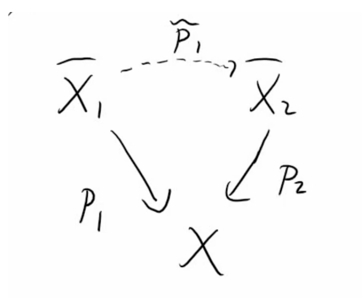
Proof:
$\underline{\implies}$
Suppose $f : (\widetilde{X_1}, \widetilde{x_1}) \to (\widetilde{X_2}, \widetilde{x_2})$ is c.s.iso. Then

$$p_1 = p_2 \circ f \implies p_{1*}(\pi_1(\tilde{X_1})) = p_{2*}(f_*(\pi_1(\tilde{X_1})))$$

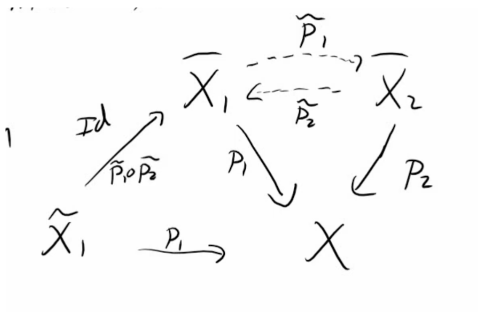
But f is a homeomorphism so f_* is an iso. So $f_*(\pi_1(\widetilde{x_1})) = \pi_1(\widetilde{x_2})$.
So,

$$p_{1*}(\pi_1(\tilde{X_1})) = p_{2*}(f_*(\pi_1(\tilde{X_1}))) = p_{2*}(\pi_1(\tilde{X_1}))$$

$\underline{\Longleftarrow}$
Going the other way, if $p_{1*}(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = p_{2*}(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$. Then by the lifting criterion, p_1 lifts to a map $\widetilde{p_1} : \widetilde{X_1} \to \widetilde{X_2}$.



Similarly, p_2 lifts to $\widetilde{p_2} : \widetilde{X_2} \to \widetilde{X_1}$. (once basepoints are chosen.) So that $\widetilde{p_1}(\widetilde{x_1}) = \widetilde{x_2}, p_2(\widetilde{x_2}) = \widetilde{p_1}(x_1)$.

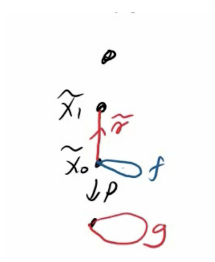


Now, $\widetilde{p_1}$ and $\widetilde{p_2}$ are continuous and also, $\widetilde{p_2} \circ \widetilde{p_1}(\widetilde{x_1}) = \widetilde{x_1}$.
But the identity $id(\widetilde{X_1})$ and $\widetilde{p_2} \circ \widetilde{p_1}(\widetilde{X_1})$ are both lifts of p_1 that agree at $\widetilde{x_1}$. So $\widetilde{p_2} \circ \widetilde{p_1} = id$ and $\widetilde{p_1} \circ \widetilde{p_2} = id$.
So $\widetilde{p_1}$ and $\widetilde{p_2}$ are inverse homeomorphisms.

$\square$

Theorem:
Let X be a p.c., l.p.c., and s.l.s.c. space. Then there is a bijection between sets of basepoint preserving isomorphism classes of covering spaces $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. (This is what we have already shown above.)
Moreover, if basepoints are ignored, then we get a bijection between isomorphism classes of p.c. covering $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Proof: We only need to prove the case without basepoints.
Let $\widetilde{x_0}$ and $\widetilde{x_1}$ be two basepoints in $p^{-1}(x_0)$ for p a covering map. Then $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x_1}$. Now, $p(\widetilde{\gamma})$ is a loop in X representing some elements $g \in \pi_1(X, x_0)$.



If f is a loop based in $\widetilde{x_0}$, then $\overline{\widetilde{\gamma}} \circ f \circ \widetilde{\gamma}$ is a loop based at $\widetilde{x_1}$.
So

$$[p(\overline{\widetilde{\gamma}} \circ f \circ \widetilde{\gamma})] = g^{-1} \circ h \circ g$$

for $h \in p_*(\pi_1(\widetilde{X}, x_0))$.
Letting $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x_i}))$ for $i \in \{0, 1\}$. We have shown that $g^{-1}H_0g \subset H_1$. (Because those loops themselves are in H_1 as it starts and ends at $\widetilde{x_1}$.) Reversing the path $\widetilde{\gamma}$ above shows $gH_1g^{-1} \subset H_0$. Conjugate this second containment by g^{-1} to get $H_1 \subset g^{-1}H_0g$.
Conversely suppose H_0 and H_1 are conjugate subgroups of $\pi_1(X, x_0)$. Let $H_1 = g^{-1}H_0g$. Let $\gamma \subset X$ be a loop representing g. Lifting g to a path in $\widetilde{X}$ starting at x_0 and repeating the argument before gives the desired result.

□

Since there is only 1 conjugacy class of trivial groups in a given group, we get the following corollary.

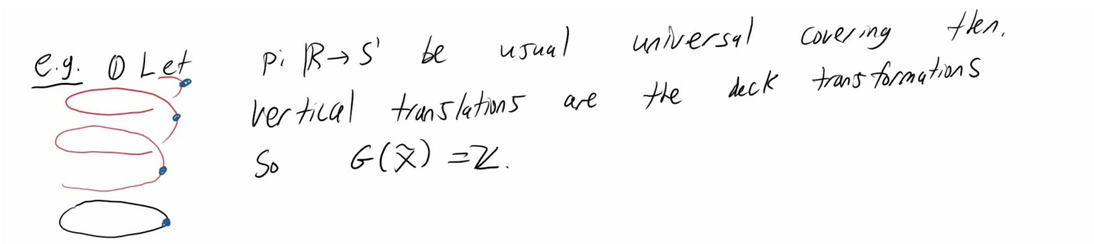
Corollary 9: Universal covering spaces are unique up to covering space isomorphism.

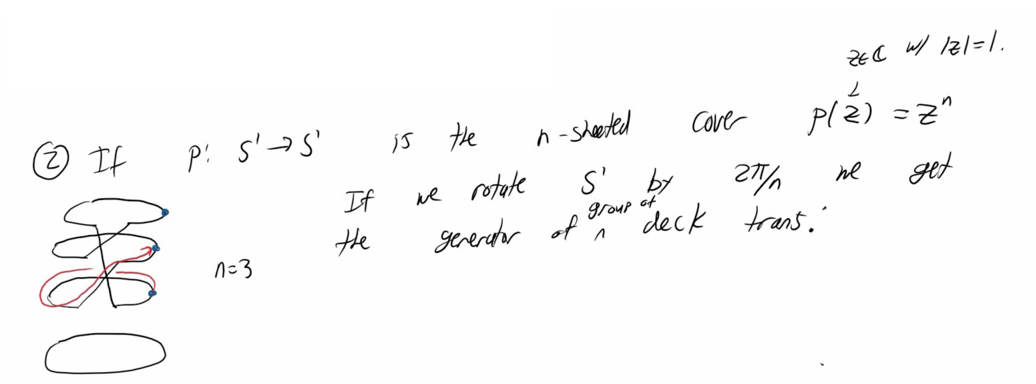
Deck Transformations

Definition: Let $p : \widetilde{X} \to X$ be a covering. A deck transformation of $\widetilde{X}$ is a covering space isomorphism $F : \widetilde{X} \to \widetilde{X}$. The group of such isomorphism is denoted $G(\widetilde{X})$. We can quickly check that this is a group.

By the unique lifting property, a deck transformation $F \in G(\widetilde{X})$ is determined by where it sends a single point. (Why is this? I am not sure.)

Examples.

- 

- 

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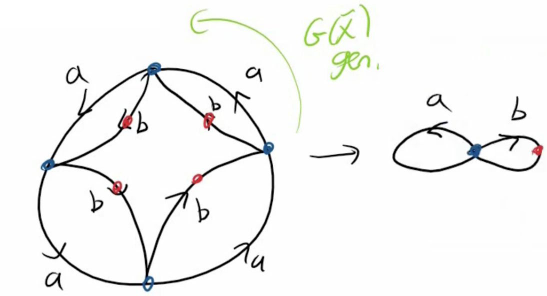
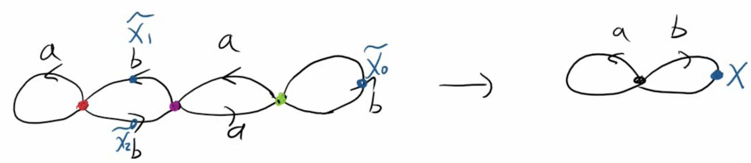


Here
$G(\widetilde{X}) = \mathbb{Z}/4\mathbb{Z}$
generated by rotating the circle by $2\pi/4$

> **Definition (Normal covering space):** A covering space $p : \widetilde{X} \to X$ is called normal if for every $x \in X$ and $\widetilde{x_0}, \widetilde{x_1} \in p^{-1}(x)$, $\exists\ F \in G(\widetilde{X})$ so that $F(\widetilde{X_0}) = \widetilde{X_1}$.

Intuitively these are the covering spaces with "most symmetry". For example, all the covering spaces above are normal covering space.
Non example is the following

Non-eg



There is a c.s iso sending $\widetilde{X_1}$ to $\widetilde{X_2}$
But there is no iso taking $\widetilde{X_0}$ to $\widetilde{X_1}$ or $\widetilde{X_2}$ (Leave out of points $\widetilde{X_1}$ $\widetilde{X_2}$ to different vertices / same vertex $\widetilde{X_0}$).

Why are the covers called normal? It's because they are related to normal subgroups.

> **Proposition:**
> Let $p : (\widetilde{X_0}, \widetilde{x_0}) \to (X, x_0)$ be path connected c.s. of a p.c., l.p.c., s.l.s.c., space X. Let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) \subset \pi_1(X, x_0)$. Then
> 1. H is normal (as a subgroup) $\iff$ p is normal (as a cover)
> 2. $G(\widetilde{X})$ is isomorphic to $N(H)/H$ where $N(H)$ is the normalizer of H in G. In particular, if p is normal, then $G(\widetilde{X}) = \pi_1(X, x_0)/H$. And for the universal cover $G(\widetilde{X}) = \pi_1(X, x_0)$.

Proof:

1. Recall that changing basepoint in cover changes the subgroup by a conjugation by $[\gamma] \in \pi_1(X)$ which lifts to a path between the two basepoints $\widetilde{x_0}, \widetilde{x_1}$.

$$[\gamma] \in N(H) \iff p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) = p_*(\pi_1(\widetilde{X}, \widetilde{x_1}))$$

$$\iff \text{The covering spaces are basepoint preserving isomorphic}$$

   That happens if one element is in the normalizer. Now, H is normal $\iff$ $N(H) = H$ $\iff$ there exists deck transformation between any two basepoint preserving isomorphisms or $p : \widetilde{X} \to X$ is normal.

2. Let $\varphi : N(H) \to G(\widetilde{X})$ set $[\gamma]$ to the deck transformation sending $\widetilde{x_0}$ to $\widetilde{x_1}$. Can check that φ is a homomorphism, and surjective. So the kernel of this map is loops in X which lifts to loops in $\widetilde{X}$. These are elements of $p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) = H$. By first isomorphism theorem, $N(H)/H = G(\widetilde{X})$.

   I dont really understand this second proof

□

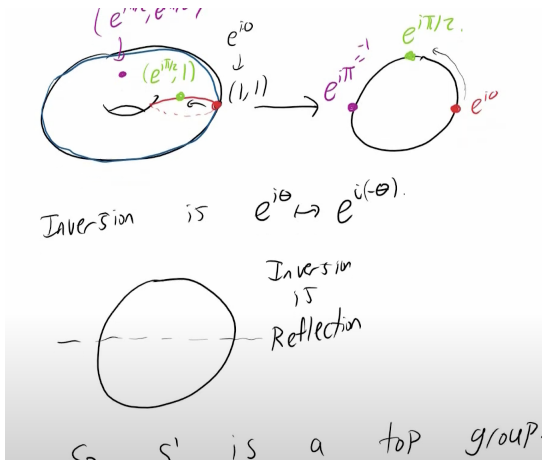
<u>Example</u>
Let $p : S^1 \to S^1$ be $p(z) = z^n$. Since $\pi_1(S^1) = \mathbb{Z}$ is abelian, every connected cover is normal. Moreover, in this case $G(\widetilde{X}) = N(H)/H = \mathbb{Z}/n\mathbb{Z}$.
I dont really understand this second proof

Lecture 13. Continuous Groups Actions

Group Actions

Definition: A topological group is a group G together with a topology on the underlying set of G, so that the maps $\cdot : G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 \cdot g_2$, and $^{-1} : G \to G$ such that $g \mapsto g^{-1}$ are continuous on the underlying topology.

Example 1:
Let $S^1 \subset \mathbb{C}$ be $\vec{z} \in \mathbb{Z}$ with $|Z| = 1$. Such numbers are of the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Then S^1 is a topological group under complex multiplication. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.
We needed that $S^1 \times S^1 \to S^1$ given by $(e^{i\theta_1}, e^{i\theta_2}) \mapsto e^{i(\theta_1+\theta_2)}$ is continuous. The product is continuous and the inversion is continuous. So S^1 is a top group.



Example 2:

Ex Let $GL(n, \mathbb{R})$ be $n \times n$ matrices w/ non-zero det.
topologized as a subspace of $\mathbb{R}^{n^2}$. $\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$

$GL(n, \mathbb{R})$ is a top group under matrix multiplication.

Example 3:

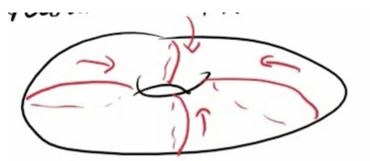
Ex Usual groups like $\mathbb{Z}$, $\mathbb{Z}_n$, D_n are top group using the discrete topology.

Continuity is given for free because of discrete topology.

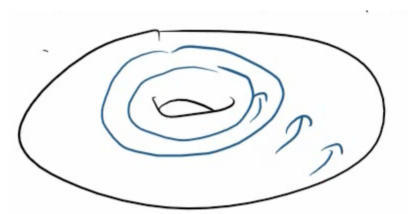
Definition: Given a top group G and a space X, a continuous G action, denoted $G \curvearrowright X$, is a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, so that

1. $e \cdot x = x$, for all $x \in X$ where $e \in G$ is identity.
2. $(gh) \cdot x = g \cdot (h \cdot x)$.

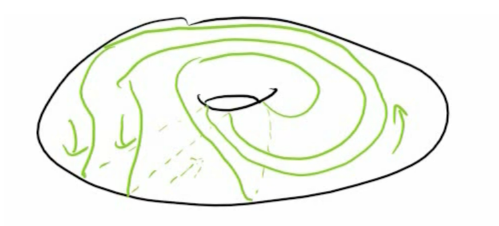
Example: Let $x \in S^n$ define $-x$ to be the antipodal point. Then $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ acts on S^n by $1 \cdot x = -x$. Check that $0 \cdot x = x, 1 \cdot (1 \cdot x) = 1 \cdot (-x) = x = (1 \cdot 1) \cdot x$.
Example: $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ acts on $T^2 = \{(x, y) \in \mathbb{C} \times \mathbb{C}, |x| = 1, |y = 1|\}$,
by $z \cdot (x, y) = (z \cdot x, y)$ which rotates around the first factor.



Or $z \cdot (x, y) = (x, z \cdot y)$, rotates around the second factor.

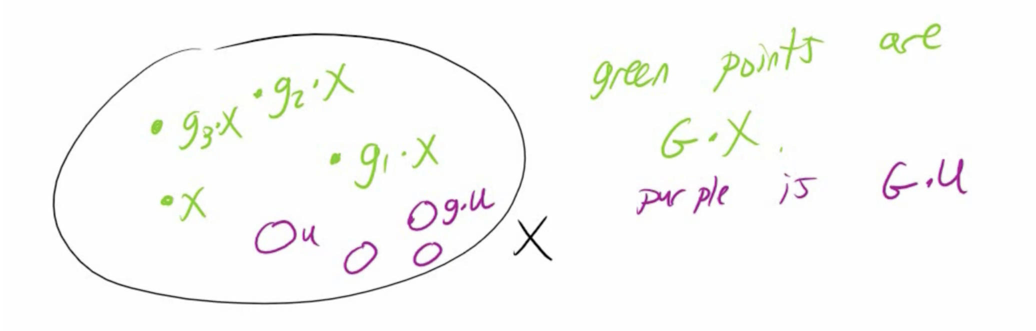


Or $z \cdot (x, y) = (z \cdot x, z \cdot y)$.



Aside: for a fixed $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism since it has a continuous inverse $x \mapsto g^{-1} \cdot x$, where $x \mapsto g \cdot x \mapsto g^{-1} \cdot (g \cdot x) = (g^{-1} \cdot g) \cdot x = e \cdot x = x$.
So a group action gives a group isomorphism from $G \to Hom(X)$.

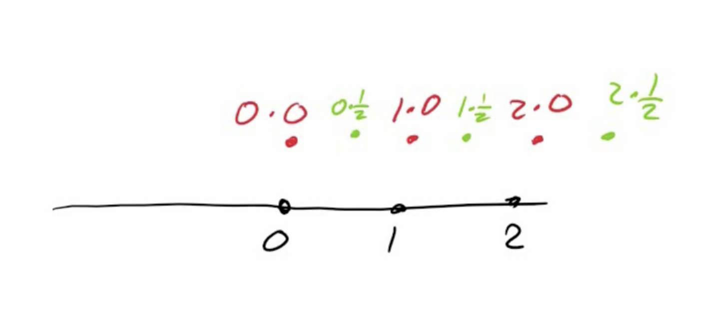
Definition: Suppose that $G \curvearrowright X$. Given $x \in X$, define the orbit of x to be $G \cdot x = \{g \cdot x \mid g \in G.\}$ Similarly, if u is a subset of X, then $G \cdot u = \bigcup_{x \in u} G \cdot x$.



We can define an equivalence relation on X by $x \sim Y$ if $G \cdot x = G \cdot y$. That is $x \sim y$ if $\exists g \in G$ so that $g \cdot x = y$.

Definition: Given a group action $G \curvearrowright X$, X/G is the quotient space $X/\sim$ where $x \sim y$ if $G \cdot x = G \cdot y$.
Note that we can call the points in X/G as $G \cdot x$ for some representative x.

Example
If $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^2$ by antipodal map $S^2/\mathbb{Z}_2 = \mathbb{R}P_2$.
Example
Let $\mathbb{Z} \curvearrowright \mathbb{R}$ by $n \cdot r = n + r$. That is, the integers form an orbit.



$\mathbb{R}/\mathbb{Z} = [0,1]/0 \sim 1 = S^1$.
Note that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $\pi_1(S^1) = \mathbb{Z}$.

Definition: An action $G \curvearrowright X$ is properly discontinuous if every $x \in X$ has a nbd u so that $u \cap g \cdot u = \varnothing$ for all $g \in G$ with $g \neq e$. Caution: there are many defns of properly discontinuous in different books.

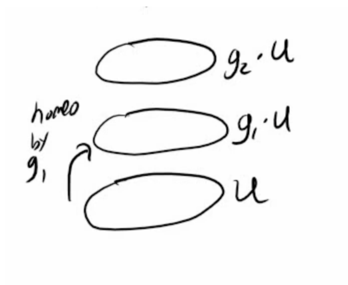
Proposition:
If $G \curvearrowright Y$ properly discontinuously with Y p.c. and l.p.c. Then

1. $p : Y \to Y/G$ given by $p(y) = G \cdot y$ is a normal covering map.
2. G is the group of deck transformations.
3. G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$.

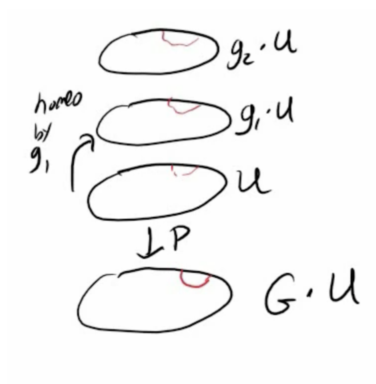
   In particular if Y is simply connected, then $\pi_1(Y/G) = G$.

Proof:
First cover Y by sets $\{u_\alpha\}$ so that $g \cdot u \cap u = \varnothing$ for all $u \in \{u_\alpha\}$ and $g \in G$ with $g \neq id$. why is this always possible?

1. For all $g \in G$, $g \cdot u \cong u$(homeo), since g^{-1} is the inverse map.

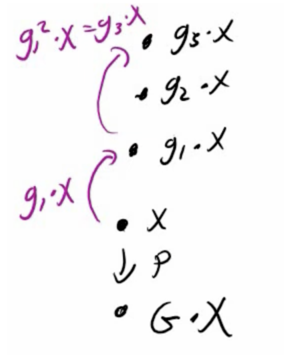
   The quotient map $p : Y \to Y/G$ simply identifies all of these sets to a single copy of $G \cdot u$, by the definition of quotient topology, $G \cdot u \cong u$.

   So p is a local homeomorphism on these sets so it is a covering map. (The properly continuous sets serve as the evenly covered neighbourhoods.)

   Why is this normal???

2. G acts by deck transformations. First, G acts by homeomorphisms. Also, it preserves fibers.

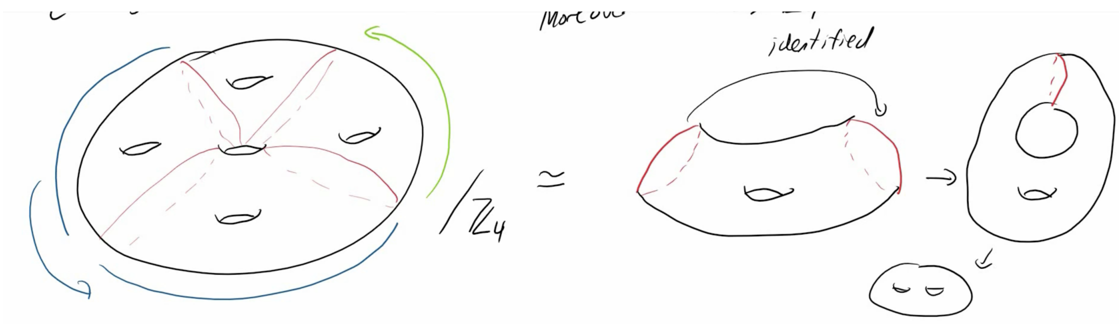


□

Finding covers

Example 1

The genus 5 surface is a 4 fold cover of the genus 2 surface. Morevoer, $\Sigma_5/\mathbb{Z}_4 = \Sigma_2$.



This tells us that $\pi_1(\Sigma_5) \trianglelefteq \pi_1(\Sigma_2)$, because it gives us a normal covering map, which implies the induced subgroup is a normal subgroup. And $\pi_1(\Sigma_2)/\pi(\Sigma_4) \cong \mathbb{Z}_4$.

Example 2

Let $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ given by $(z_1, x_2) \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1$. Complex multiplication is a great way to generate group actions. $\mathbb{Z}_p$ acts on S^3 properly discontinously by

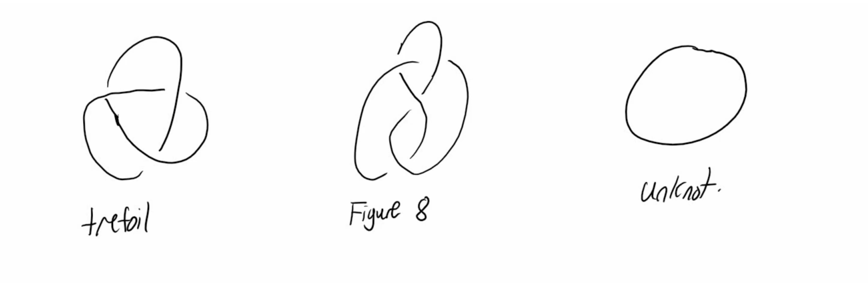
$$[q] \cdot (z_1, z_2) = (e^{2\pi i \cdot q/p} z_1, e^{2\pi i \cdot q/p} z_2)$$

The space $S^3/\mathbb{Z}_p$ is called the Lens space $L(p, q)$.

Interesting fact, the space $L(7, 1), L(7, 2)$ are homotopy equivalent but not homeomorphic.

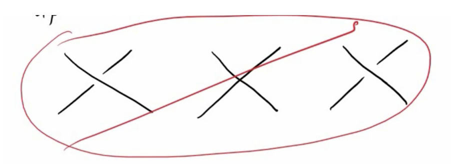
Lecture 14 Knots and the Wirtinger Presentation

Intuitively, a knot is a closed piece of string in the 3 space.
Are mirror images of knots equivalent?



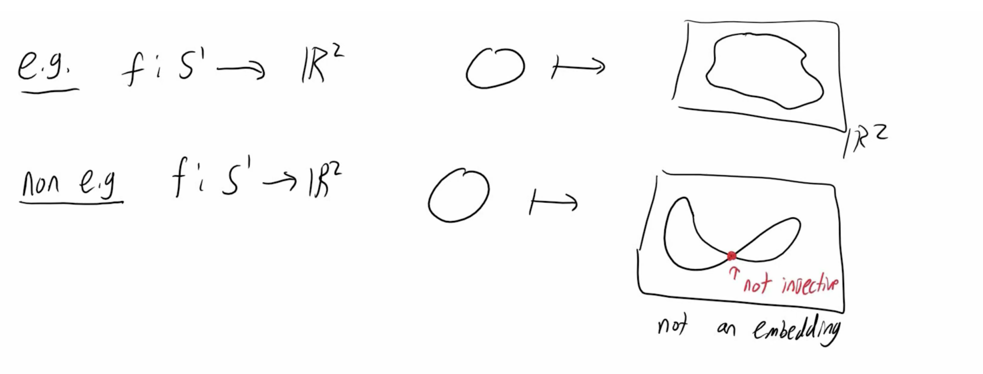
Knots are considered up to continuous deformations but no passing through itself.

Definition 0.27 (Embedding): Let X and Y be topological spaces then an embedding of X onto Y is a map $f : X \to Y$ which is a homeomorphism onto its image.



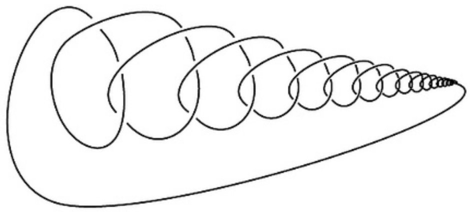
Note that the part of the crossing is NOT injective. So it is not a homeomorphism.

Definition 0.28 (Isotopy): An isotopy of a topological space X is a map $F : X \times I \to X$ so that for each $t \in I$, $f\,|_{X\times\{t\}}: X \times \{t\} \to X$ is a homeomorphism.



is it more strict than a homotopy because at each slice, it needs to be homeomorphic to the original set.

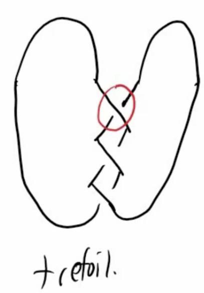
Definition 0.29 (Knot): A knot is an embedding $f : S^1 \to \mathbb{R}^3$ which can locally be approximated by a line. We call $f(S^1)$ the knot. The reason why we use "Approximated by a line" is because we want to rule out knots like the below: (the wild knot)



At the limit point, there is no line that can approximate it.

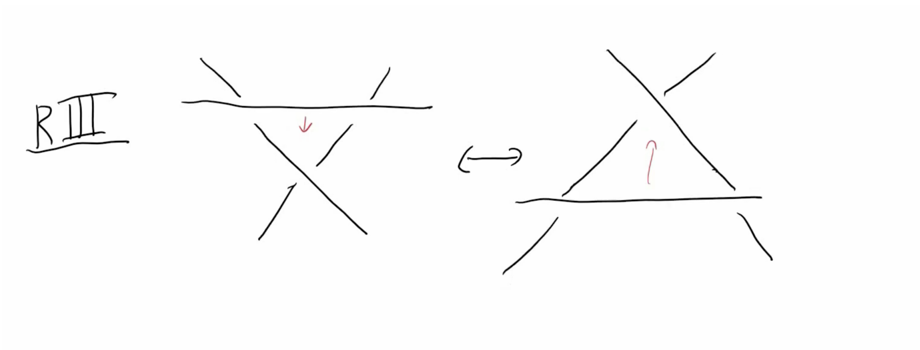
Definition 0.30 (Equivalent knots): Two knots k_1, k_2 are equivalent if there exists an isotopy $F : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ that takes k_1 to k_2.

In fact, we can squish a knot down to $\mathbb{R}^2 \times [-\epsilon, \epsilon]$.
So we can encode a knot in $\mathbb{R}^2$ using a knot diagram which is a 4−valent (4 edges at each vertex) with over/under crossing information. Manipulating these objects is easier than manipulating objects in 3 space.

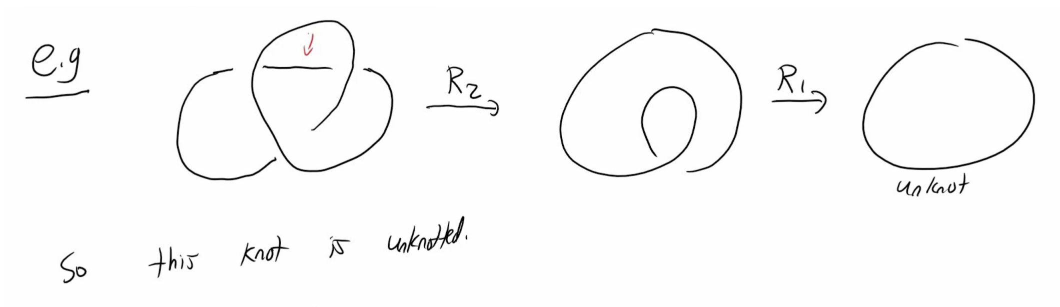


Reidemeiter moves

There are three Reidemeister moves.



These three are all we need!
Since these moves correspond to isotopies. Any two diagrams related by these moves represent equivalent knots.



Theorem 0.28 ((Reidemeister, Alexander, Briggs)):
Two knots k_1, k_2 are equivalent $\iff$ they have diagrams related by $RI, RII, RIII$ and planar isotopy. (wiggle a connected line.)

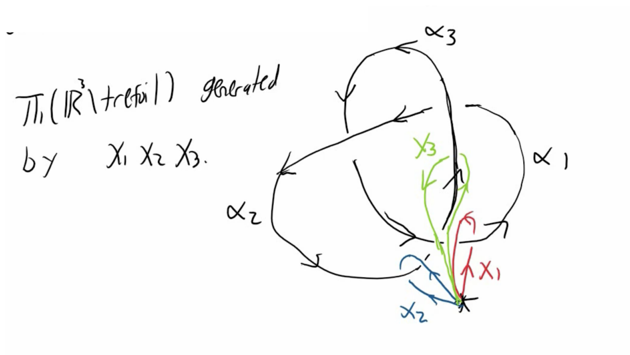
Problems:

1. Given two knots, which are equivalent, how many moves do you need to do to get between them? (A lot, it can be arbitrarily big.)
2. If k_1, k_2 are equivalent knots, then $\mathbb{R}^3 \setminus K_1$, and $\mathbb{R}^3 \setminus K_2$ are homeomorphic. So if we can distinguish the complements, we can distinguish the knots.

   Our favourite way of distinguishing spaces is π_1. So now we need to figure out ways to compute $\pi_1(\mathbb{R}^3 \setminus K_i)$.

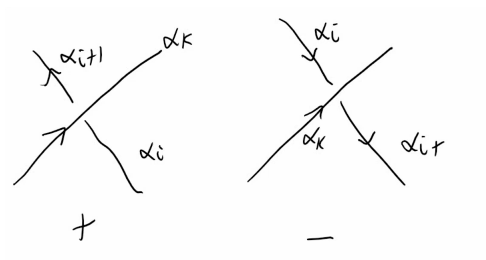
0.4 Wirtinger Presentation

Wirtinger presentation takes in a knot diagram and outputs a group presentation for $\pi_1(\mathbb{R}^3 \setminus K)$. Small amount of auxiliary data is an orientation of the knot, but this won't affect the group.
So, we need generators and relations.
Generators: place a basepoint "above" the diagram, and draw loops arouund each connected components of diagram using right hand rule. (imaging you're holding onto the coil, and your thumb point towards directionn, and the directions follow from your lower arms onto your finger).

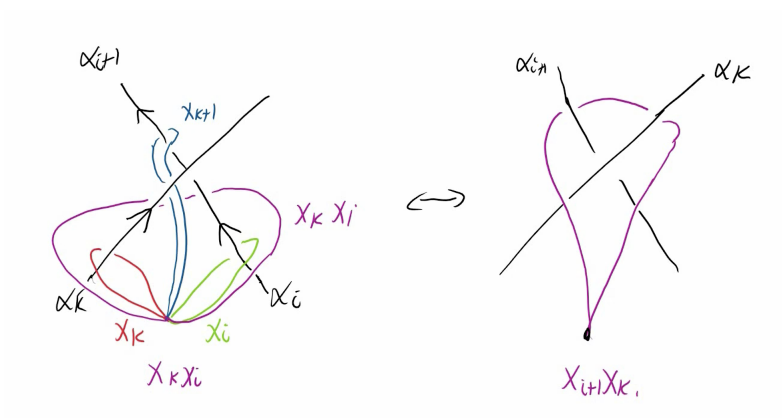


Relations: we can assign a sign to each oriented crossing by the following. Take the top strand and rotate it counterclockwise until it aligns with bottom strand.

- if arrow matches then $+$
- if arrow clash then $-$



Zoom in on a $+$ crossing, we can see a relation.



So $x_k x_i = x_{i+1} x_k \implies x_{i+1} = x_k x_i x_k^{-1}$.
Similarly, at a negative crossing, we have $x_{i+1} = x_k^{-1} x_i x_k$.
It turns out that these are all the relations!

Theorem 0.29:
Let $K \subset \mathbb{R}^3$ be a knot and let D be a diagram for k with n strands, $\alpha_1, \ldots, \alpha_n$. Let x_i be the loop around α_i and let r_i be the relation at each crossing given by

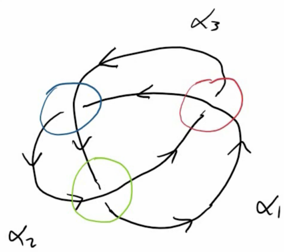
$$x_{i+1} = x_k x_i x_k^{-1} \text{ at + crossing}$$

$$x_{i+1} = x_k^{-1} x_i x_k \text{ at - crossing}$$

then
$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$. Moreover, any one relation can be omitted and this is still a presentation for $\pi_1(\mathbb{R}^3 \setminus K)$.

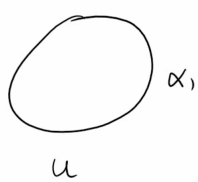
<u>Examples</u>

e.g.



$$\pi_1(\mathbb{R}^3 \setminus K) = \left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_3 = x_1^{-1} x_2 x_1 \\ x_2 = x_3^{-1} x_1 x_3 \\ x_1 = x_2^{-1} x_3 x_2. \end{array} \right\rangle$$

e.g



$$\pi_1(\mathbb{R}^3 \setminus u) = \langle x_1 \mid \quad \rangle. = \mathbb{Z}.$$

Given these two presentations, how do we know it's actually the same?
Note that we can throw out a relation, and get rid of a generator because one can be written in terms of the two others.

$$\pi_1(\mathbb{R}^3 \setminus K) = \left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} x_3 = x_1^{-1} x_2 x_1 \\ \downarrow \\ x_1 = x_2^{-1} x_3 x_2. \end{array} \right\rangle$$

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, x_2 \mid x_1 = x_2^{-1} x_1^{-1} x_2 x_1 x_2 \rangle =$$
$$\langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle.$$

p, q are still generators of the group!

Let $p = x_1 x_2$ and $q = x_1 x_2 x_1$. Note $x_1 = p^{-1} q$

$x_2 = x_1^{-1}(x_1 x_2) = q^{-1} p \cdot p = q^{-1} p^2$.

So p and q generate and the relation

$$x_1 x_2 x_1 = x_2 x_1 x_2 \implies \underbrace{(x_1 x_2 x_1)}_{q} \cdot \underbrace{(x_1 x_2 x_1)}_{q} = \underbrace{(x_1 x_2)}_{p} \cdot \underbrace{(x_1 x_2)}_{p} \cdot \underbrace{(x_1 x_2)}_{p}$$

So $\pi_1(\mathbb{R}^3 \setminus K) = \langle p, q \mid p^3 = q^2 \rangle$.

This looks a lot like $D_3 = \langle r, s \mid r^3 = s^2 = 1, rs = sr^2 \rangle$

There is a surjective hom $f : \pi_1(\mathbb{R}^3 \setminus K) \to D_3$ by

$f(p) = r$ and $f(q) = s$.

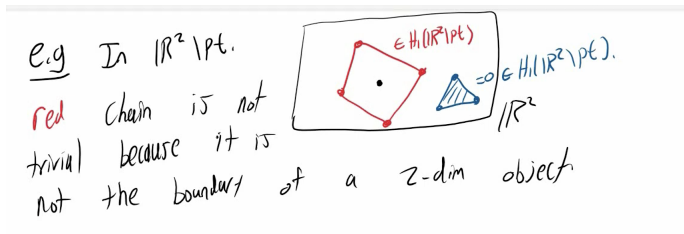
1. D_3 is not abelian ($rs = sr^2$)
2. any map out of an abelian group has an abelian image.
3. $\pi_1(\mathbb{R}^3 \setminus U) = \mathbb{Z}$ is abelian.

This finally proves that the trefoil is not equivalent to the unknot.

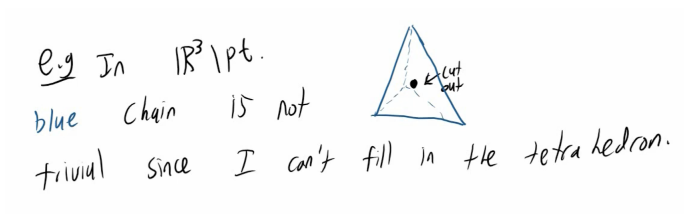
Lecture 15. Simplicial Homology

Intuition

Problem with π_1 is that they only see low dimensional information. That is, if X is a CW complex, then $\pi_1(X) = \pi_1(X^2)$ where X^2 is the 2 skeleton.
There exist groups π_k for $k \geq 2$ but they are difficult to compute.
We will define homology groups which come in degrees $H_0(X), H_1(X), H_2(X)$. These are abelian groups.
Roughly, $H_i(X)$ sees $i-$dimensional holes in the space. Namely, $H_i(X)$ detects i-dimensional objects which do not bound $i+1$-dimensional objects.



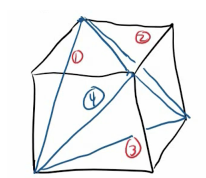
In here, $H_1(\mathbb{R}^2 \setminus \mathrm{pt}) = \mathbb{Z}$



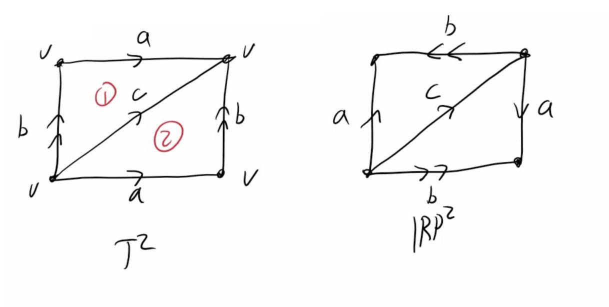
In here, $H_1(\mathbb{R}^3 \setminus \mathrm{pt}) = \mathbb{Z}$

Δ-complexes

Recall that polygons can be cut into triangles. Similarly, solids can be cut into tetrahedra.



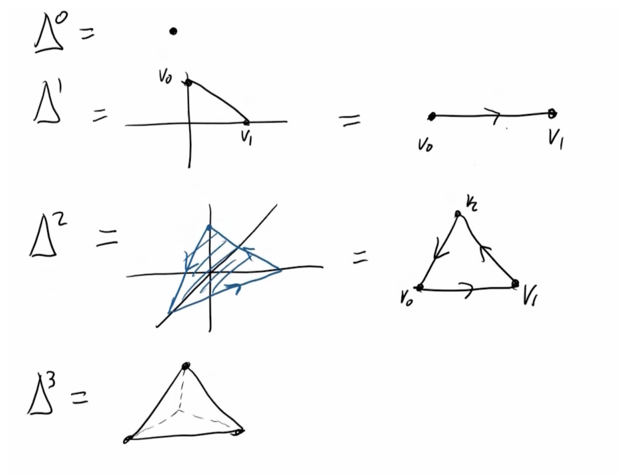
Similarly, this works on more intricate shapes.

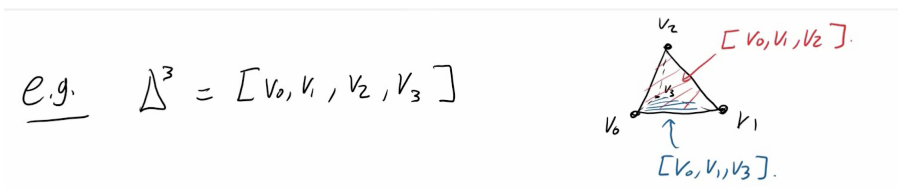


Simplices

Definition (Standard n simplex):
A standard n-simplex $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0, \forall i\}$.

We usually write this as $\Delta^n = [v_0, v_1, v_2 \ldots v_n]$, where the square bracket [] represents the convex hull. Deleting a vertex and filling in the convex hull of an $n-$simplex leaves an $n-1$ simplex.





Definition (Boundary, faces, open simplex.): Given a simplex $\Delta^n = [v_0, \ldots, v_n]$, the simplices $[v_0, \ldots, \hat{v}_i, \ldots v_n]$ (the $\hat{}$ means omit) are call the faces. The union of all the faces is called the boundary of Δ^n. The open simplex is the space $\Delta^n - \partial\Delta^n$ and is denoted Δ^{on}.

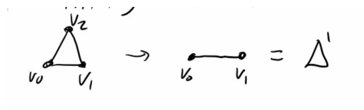
Definition (Δ-complex structure): A $\Delta-$complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ with n depending on α, such that

1. The restriction $\sigma_\alpha |_{\Delta^{on}}$ is injective and for each $x \in X$, x is in the image of some $\sigma_\alpha |_{\Delta^{on}}$. (This prevents squashing everything down into a point.)
2. Each restriction of σ_α to a fact of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. (This ensures that the simplices are glued along the correct corresponding faces.)
3. A set $A \subset X$ is open iff $\sigma^{-1}(A)$ is open in Δ^n for each σ_α.

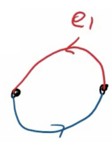
If a space has a Δ-complex structure, then it is obtained by gluing togehter simplices.
All CW complexes have a $\Delta-$complex structure.

Chains and boundaries

Definition (Chains): Let X be a $\Delta-$complex. We define $\Delta^n(X)$ to be the free abelian group generated by $n-$dim simplices on X. Elements of $\Delta^n(X)$ are called $n-$chains.



e.g.



then $3e_1 - 2e_2 \in \Delta^1(S^1)$.

$\Delta^1(S^1) = \mathbb{Z} \oplus \mathbb{Z}$

Elements of $\Delta^n(X)$ can be written as $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$. Recall faces are of the form $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. We want to orient everything so that $\partial\Delta^n$ is an $n-1$chain.

$\partial[v_0, v_1] = [v_1] - [v_0]$.

$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

Definition 0.31 ($\partial\sigma_\alpha$): If $\sigma_\alpha : \Delta^n \to X$ be an $n-$simplex in X, then we define

$$\partial\sigma_\alpha = \sum_{i=0}^{n}(-1)^i\sigma_\alpha \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

If $\sum_\alpha n_\alpha\sigma_\alpha$ is an $n-$chain, then define

$$\partial\sum_\alpha n_\alpha\sigma_\alpha = \sum_\alpha n_\alpha\partial\sigma_\alpha$$

Lemma 1 *Short exact sequences*

Lemma The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is the zero map ($\partial^2 = 0$).

So basically

$$\partial^2 = 0.$$

Note that this literally means the Δ_{n-1} map restricted to one of the boundaries of the Δ complex structure. □

Proof: We prove it for a basis element $\sigma \in \Delta_n(X)$.

$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma_i \mid_{[v_0, v_1, \ldots, \hat{v_i}, \ldots, v_n]}$$

so $\partial_{n-1}\partial_n$ breaks up.

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma \mid_{[v_0, \ldots, \hat{v_j} \ldots \hat{v_i} \ldots v_n]}$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \mid_{[v_0, \ldots, \hat{v_i} \ldots \hat{v_j} \ldots v_n]}$$

Note that every term in the first line also appear in second line with opposite sign so $\partial_{n-1}\partial_n(\sigma) = 0$.

□

Simplicial Homology

Definition 0.32 (Chain complex): We now have a setup

$$\Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \ldots$$

with $\partial^2 = 0$.
This is called a chain complex.
In particular, $Im(\partial_{n+1}) \subset Ker(\partial_n)$. All groups are abelian so we can take quotients.

Definition 0.33 (nth simplical homology group): The nth simplical homology group of a $\Delta-$complex X is the group $H_n^{\Delta}(X) = ker(\partial_n)/Im(\partial_{n+1})$. Note that this makes sense. That is, every thing in the image of ∂_n is in the kernel of ∂_n by the previous lemma. Also the group is abelian, we have normal subgroups so we can take quotients.
$Ker(\partial_n)$ are called <u>n-cycle</u>. $Im(\partial_{n+1})$ are called <u>boundaries</u> of other objects.
Intuitively, cycles are objects with no boundary. So we are measuring is objects with no boundaries, which are also not the boundaries of other objects.

<u>Example 1</u>
Let S^1 be built with one copy of Δ^o and one copy of Δ^1.

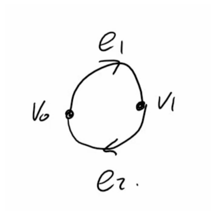
$\Delta_0 =$ Free ab gp gen by 0-simp=free ab gp on $V_0 = \mathbb{Z}$.
$\Delta_1 =$ Free ab gp gen by $e_1 = \mathbb{Z}$.
$\Delta_2 = \Delta_3 = \ldots = 0$
So ∂_0 is always 0 so ker $\Delta_0 = \mathbb{Z}$ generated by v_0.
$\partial_1(e_1) = [v_0] - [v_0] = 0$ so $Im(\partial_1) = 0 \subset \Delta_0$. $Ker(\partial_1) = \mathbb{Z}$ generated by e_1. $\partial_2 = 0$.
So

- $H_0^\Delta(S^1) = Ker\partial_0/Im\partial_1 = \mathbb{Z}$
- $H_1^\Delta(S^1) = Ker\partial_1/Im\partial_2 = \mathbb{Z}$
- $H_2^\Delta(S^1) = H_3^\Delta(S^1) = \ldots = 0$

Another way to write this is:

$$H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & o.w. \end{cases}$$

Example 2 What if we build the S^1 in another way?



We built S^1 as above.
So

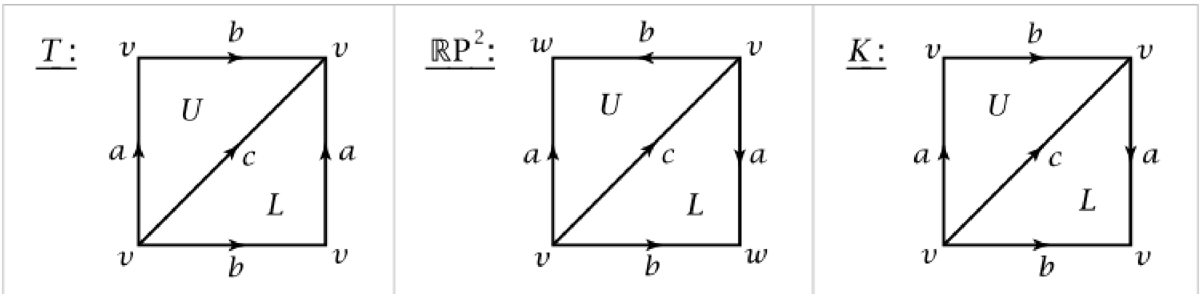
- $\Delta_0(S^1) = \mathbb{Z} \oplus \mathbb{Z}$, $\Delta_1(S^1) = \mathbb{Z} \oplus \mathbb{Z}$
- $\partial_1(e_1) = [v_1] - [v_0], \partial_1(e_2) = [v_0] - [v_1]$, so $\partial(e_1 + e_2) = 0$.

So Kernel (∂_1) is $1d$ generated by $e_1 + e_2$. $H_1^\Delta(S^1) = \mathbb{Z}$, which is the free ab gp generated by $e_1 + e_2$.

- $Ker(\partial_0) = \mathbb{Z} \oplus \mathbb{Z}$
- $Im(\partial_1) = span(1, -1)$, where 1 is v_0, -1 is v_1
- So $H_0^\Delta(S^1) = \mathbb{Z}$.

Example 3
Recall that $S^n = D^n \cup D^n$, former is the lower hemi, latter is the upper hemi. Similar to previous constructions, $S^n = A \cup B$ with $A, B = \Delta^n$.
Then $\Delta_n(S^1) = \mathbb{Z} \oplus \mathbb{Z}$ and $Ker(\partial_n) = A + B$. So $H_n(S^n) = \mathbb{Z}$. And $H_{n+l}(S^n) = 0$ for $k \geq 1$. This means that $S^n \neq S^m$ for $n \geq m$.

Simplicial homology groups of Torus, RP2 and K



Torus

<u>Simplices</u>

- $C_0 = \mathbb{Z}$, generated by v
- $C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, generated by a, b, c
- $C_2 = \mathbb{Z} \oplus \mathbb{Z}$, generated by U, L
- for $n \geq 2, C_n = 0$

<u>Boundary maps</u>

- $\partial_0 = 0$ by definition
- $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$
- $\partial_2(U) = \partial_2(L) = a + b - c$ (Since abelian, we can move generators around)

<u>Homology groups</u>

- $H_0^\Delta = ker(\partial_0)/Im(\partial_1)$

  Note that $ker(\partial_0) = C_0 = \mathbb{Z}$ and $Im(\partial_1) = 0$.

  So $H_0^\Delta = ker(\partial_0)/Im(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}$.

- $H_1^\Delta = ker(\partial_1)/Im(\partial_2)$

  $ker(\partial_1)$ is generated by a, b, c, so $ker(\partial_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and $Im(\partial_2)$ is generated by multiplies of $a + b - c$. So $Im(\partial_2) = \mathbb{Z}$.

  Hence $H_1^\Delta = ker(\partial_1)/Im(\partial_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$.

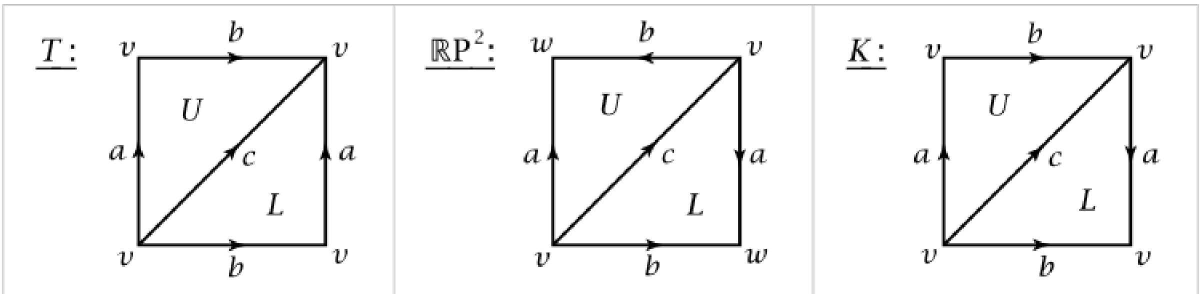
- $H_2^\Delta = ker(\partial_2)/Im(\partial_3)$

  Note that $Im(\partial_3) = 0$ because $C_3 = 0$.

  $ker(\partial_2) = \mathbb{Z}$. Say $\partial(pU + qL) = (p+q)(a + b - c) = 0 \iff p = -q$, so kernel generated by $(U - L)$.

  So $H_2^\Delta = ker(\partial_2)/Im(\partial_3) = \mathbb{Z}/0 = \mathbb{Z}$.

- For $n \geq 2$, $H_n^\Delta = ker(\partial_n)/Im(\partial_{n+1})$, we know C_n is empty for $n \geq 2$. So It is always $H_n^\Delta = 0$.



Projective plane

Simplices

- $C_0 = \mathbb{Z} \oplus \mathbb{Z}$, generated by w, v.
- $C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, generated by a, b, c.
- $C_2 = \mathbb{Z} \oplus \mathbb{Z}$, generated by U, L.
- for $n \geq 2, C_n = 0$

Boundary maps

- $\partial_0(w) = \partial_0(v) = 0$ by definition.
- $\partial_1(a) = \partial_1(b) = (w - v), \partial_1(c) = v - v = 0$.
- $\partial_2(U) = c + b - a, \partial_2(L) = a - b + c$.

Homology groups

- $H_0^{\Delta} = ker(\partial_0)/Im(\partial_1)$

  So $ker(\partial_0) = \mathbb{Z} \oplus \mathbb{Z}$ by definition. $Im(\partial_1)$ is generated by $w - v$, so equal to $\mathbb{Z}$. So $H_0^{\Delta} = ker(\partial_0)/Im(\partial_1) = \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} = \mathbb{Z}$.

- $H_1^{\Delta} = ker(\partial_1)/Im(\partial_2)$

  Note that $\partial_1(qa + pb) = (q + p)(w - v)$, so kernel is generated by $\langle b - a, c \rangle$. $Im(\partial_2)$ is generated by $c + b - a, a - b + c$.

  So we have $\{b - a, c\}/\{c + b - a, a - b + c\} = \{b - a, c\}/\{2c, a - b + c\} = \{a - b + c, c\}/\{2c, a - b + c\} = \mathbb{Z}_2$.

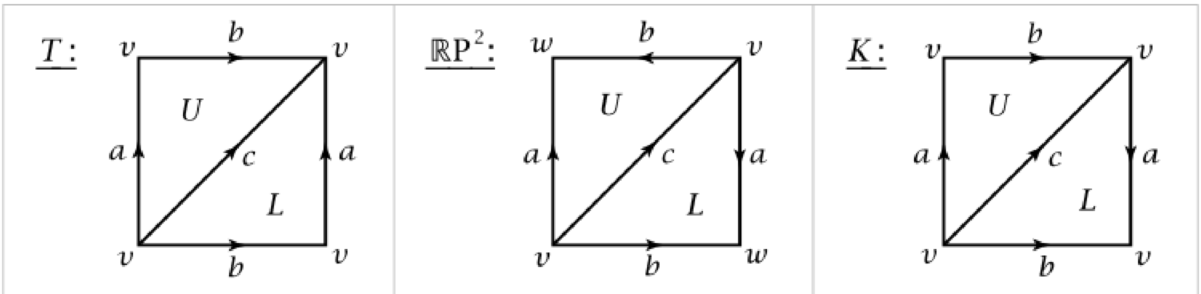
  Therefore $H_1^{\Delta} = ker(\partial_1)/Im(\partial_2) = \mathbb{Z}_2$.

- $H_2^{\Delta} = ker(\partial_2)/Im(\partial_3)$

  $\partial_1(pU + qL) = p(c + b - a) + q(a - b + c)$. This is 0 iff $p = q = 0$. So the kernel is 0.

  Hence $H_2^{\Delta} = ker(\partial_2)/Im(\partial_3) = 0$.

- For $n \geq 2$, $H_n^{\Delta} = ker(\partial_n)/Im(\partial_{n+1}) = 0$.



K

Simplices

- $C_0 = \mathbb{Z}$, generated by v
- $C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, generated by a, b, c
- $C_2 = \mathbb{Z} \oplus \mathbb{Z}$, generated by U, L
- for $n \geq 2, C_n = 0$

Boundary maps

- $\partial_0(v) = 0$ by definition
- $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$
- $\partial_2(U) = a + b - c, \partial_2(L) = a - b + c$

Homology groups

- $H_0^\Delta = ker(\partial_0)/Im(\partial_1)$
  $ker(\partial_0) = \mathbb{Z}$ by definition. $\text{Im}(\partial_1) = 0$. So $H_0^\Delta = ker(\partial_0)/Im(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}$.
- $H_1^\Delta = ker(\partial_1)/Im(\partial_2)$
  $ker(\partial_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ since ∂_1 sends everything to 0. Also, $Im(\partial_2)$ is generated by $a + b - c, a - b + c$.
  Then $\langle a, b, c\rangle/\langle a+b-c, a-b+c\rangle = \langle a+b-c, b, c\rangle/\langle a+b-c, 2b-2c\rangle = \langle b, c\rangle/\langle 2b-2c\rangle = \langle b-c, c\rangle/\langle 2b-2c\rangle = \langle d, c\rangle/\langle 2d\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}$.
  So $H_1^\Delta = \mathbb{Z}_2 \oplus \mathbb{Z}$.
- $H_2^\Delta = ker(\partial_2)/Im(\partial_3)$
  $ker(\partial_2) = 0$ because $\partial_2(pU + qL) = p(a + b - c) + q(a - b + c) = 0$ iff $p = q = 0$.
  So $H_2^\Delta = 0$.
- For $n \geq 2$, $H_n^\Delta = ker(\partial_n)/Im(\partial_{n+1}) = 0$.

March 9th office hours

Some intuitions

Simplicial complexes vs Δ complexes

- Simplicial complexes are very complicated where the Δ complexes are easier to work with. For example, simplicial complexes in torus requries 10 generators (10×10 matrix reductions) whereas the Δ complexes only is 3 generators.

Galois correspondence of covering spaces

- Deck transformations fix the important parts. So the deck transformations relate to the group automorphisms which relates to galois correspondences.

Homology measures what property of two maps

- Homology measures how two maps fit into short exact sequences. Or how the image of one fits into the kernel of anther one. Whether the kernel fits inside the image, and so we observe the quotients.

How homology measures holes

- Recall how if we have a triangle missing a hole, we cannot fit D^2 in the interior of that triangle. So in some sense, we wont have the image map of some Δ_2 into this boundary. So we can think of it as one less copy of $\mathbb{Z}$ to be moded out, i.e. one less $\mathbb{Z}$ in the quotient, so one more copy of $\mathbb{Z}$ in the homology group. Speaking of copies is too generalized, but you get the idea why it's called "measuring holes".

  For example, $H_1(\mathbb{R}^n - pt_1, pt_2, \ldots, pt - k) = \mathbb{Z}^k$.

Why do elements commute

- Because the C_n are generated by generators, so they are the free groups $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, hence commutative.

Lecture 16

Last time, we learned about Δ complexes- simplicial decompositions of spaces. We defined $\Delta_n(X)$ the free abelian group generated by simplices $\partial[v_0, \dots, v_n] = \sum_{i=1}^{n}(-1)^i[v_0, \dots, \hat{v_i}, \dots, v_n] \in \Delta_{n-1}(X)$.
The maps $\partial_n : \Delta_n \to \Delta_{n-1}$ had the property that $\partial_{n-1} \circ \partial_n = 0$.

Definition (Chain complex (of Abelian groups)): The chain complex $(C., \partial.)$ is a collection of abelian groups/maps

$$\dots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

so that $\partial_{i-1} \circ \partial_i = 0$ for all i.

In a chain complex, we have $im(\partial_{i+1}) \subset Ker(\partial_i)$ so we may define the following:

Definition 0.34: Given a chain complex $(C., \partial.)$, we define $H_n(C) = Ker(\partial_n)/Im(\partial_{n+1})$.

Singular Homology

Definition 0.35: A singular n-complex is a map $\partial : \Delta^n \to X$, here singular is said in the sense of singularities. That is, we don't have the same descriptions as in Δ complexes. Also the singular complexes can "fold up" on itself, and it's still okay. (But Δ_n cannot fold on itself.)
Let $C_n(X)$ be the free abelian group generated by all of these maps $(\sigma : \Delta^n \to X)$, we call elements of $C_n(X)$ (singular) $n - chains$,. We have the same ∂ mas as before, which on the basis, is defined by

$$\partial_\sigma = \sum_{i=0}^{n}(-1)^i[v_0 \dots \hat{v_i} \dots v_n]$$

Some proofs as before shows that $\partial^2 = 0$. So we get a chain complex

$$\dots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

Definition 0.36: The singular homology groups of a space X are defined to be $H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$ where

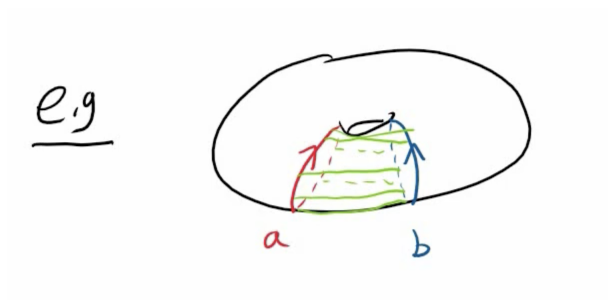
$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}}$$

is the group of singular n chains on X.

Lemma 0.30:
If X and Y are homeomorphic then $H_n(X) = H_n(Y)$ for all of n.

Proof: Proof idea: If $\sigma : \Delta^n \to X$ is an n-chain and $f : X \to Y$ is a homeo, then $f \circ \sigma : \Delta^n \to Y$ is a singular n chain on Y.
Also $\partial(f \circ \sigma) = f \circ \partial\sigma$, so all elements in Ker/Im goes to Ker/Im. □

Definition 0.37: $a, b \in C_n(x)$ are homologous if $a = b \in H_n(X)$. Note that $a = b \iff a - b = 0 \in H_n(x) \implies a - b = 0 \in Ker(\partial_n)/Im(\partial_{n+1})$. So $a - b \in Im(\partial_{n+1})$.

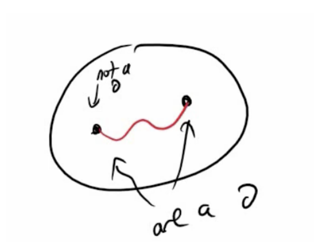


Since a and b cobound an annulus, they are homologous.

Properties of Singular Homology

Lemma:
If X is path connected then $H_0(X) = ker(\partial_0)/Im(\partial_1) = C_0/Im(\partial_1) = \mathbb{Z}$.

Proof: The proof idea:
Note that any one point is not a boundary, but if you have two points, then the two points is a boundary of the path that connects eachother. So, each points are homologues to any one point.



The proof:
Define a map $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. This is surjective and we claim that $ker(\epsilon) = Im(\partial_1)$.
To show that $Im(\partial_1) \subset Ker(\epsilon)$: Let $C_1 = \sum_i n_i\sigma_i^1$ be a 1 diml simplex. Then we will show its boundary lies inside the kernel of ϵ. Then

$$\epsilon(\partial_1 c_1) = \epsilon\left(\sum_i n_i v_i^1 - \sum_i n_i v_i^0\right) = \sum_i n_i - \sum_i n_i = 0$$

note that the 1 index means terminal vertex of σ_i and the v_i^0 means the initial vertex.

To show that $Ker(\epsilon) \subset Im(\partial_1)$: if $c_0 \in \sum_i n_i x_i$ has $\epsilon(c_0) = 0$, then $\sum_i n_i = 0$. Since X is path connected, we can fix a basepoint b and take paths from b to each x_i which we call τ_i. Let $C_1 = \sum_i n_i \tau_i$.
Then $\partial c_1 = \sum_i n_i x_i - \sum_i n_i b$. Since $\sum_i n_i = 0$, we have $\partial c_1 = \sum_i n_i x_i$, so we have found that for any c_0 that ϵ maps to 0, we find some C_1 with boundary $\partial c_1 = c_0$. so $ker(\epsilon) = Im(\partial_1)$ and $H_0(X) = C_0(X)/Im(\partial_1) = C_0(X)/ker(\epsilon)$. Now, what is $ker(\epsilon)$? it is zero. So we know that $C_0(X)$ is surjective, and it mod the kernel is 0: $C_0(X)/ker(\epsilon) = C_0(X)/0 = \mathbb{Z}/0$.
Ask why this is $\mathbb{Z}$?? ☐

Proposition 0.31:
If X has path components $X_1, \ldots X_k$, then $H_n(X) = \oplus H_n(X_i)$.

Proof: The proof idea is since Δ^n is connected, so is $\sigma_i : \Delta^n \to X$, so each map lies in 1 connected component. Also if $\Sigma^n \subset X_i$, so is $\partial\Delta^n$. So the components do not interact. ☐

Corollary 10: $H_0(X) = \mathbb{Z}^k$ where k is the number of components.

Induced maps

If $f : X \to Y$ is a map, we get a map $f_\# : C_n(X) \to C_n(Y)$, on a basis.

This is given by $\sigma : \Delta^n \to X \mapsto f \circ \delta : \Delta^n \to Y$.
So if $C = \sum_i n_i \sigma_i \in C_n(X)$. Then $f_\#(c) = \sum_i n_i f \circ \sigma_i \in C_n(Y)$.
Now

$$f_\#(\partial\sigma) = f_\#(\sum_i (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i \ldots, v])$$

$$= \sum_i (-1)^i f_\#(\sigma \mid [v_0 \ldots \hat{v}_i \ldots v_n]) = \sum_i (-1)^i f \circ \sigma \mid [v_0, \ldots \hat{v}_i \ldots v_n] = \partial f_\#(\sigma)$$

So $\partial f_\# = f_\# \partial$.
We get the following commutative diagram.

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
 & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Definition 0.38: If (C_n, ∂^c) and (D_n, ∂^D) are chain complexes, then a chain map is a collection of maps of the form $f : C_n \to D_n$ with $f(C_i) = D_i$ and $f \circ \partial^C = \partial^D \circ f$.

Proposition 0.32:
If $f : (C., \partial^C) \to (D., \partial^D)$ is a chain map, then f induces a map $f_* : H_n(C.) \to H_n(D.)$.

Proof:
In short words, f maps things from kernel of partial C into kernel of partial D. It also maps images in partial c to images of some other elements in partial D. Therefore, the elements there makes sense.
If $c \in C_n$ is a cycle, $\partial_c = 0$, then

$$\partial^D f(c) = f(\partial^C c) = f(0) = 0$$

so $f(c) \in Ker(\partial^D)$.
Similarly, if $b \in C_n$ is a boundary with $\partial^C a = b$, then

$$f(b) = f(\partial^c a) = \partial^D f(a)$$

so $f(b)$ is a boundary.
Then if $h \in H_n(c) = ker(\partial_n^c)/Im(\partial_{n+1}^c)$ is represented as $c + b$ where c is cycle, b is boundary, then $f(h) = f(c) + f(b)$ where the $f(c) \in Ker(\partial_n^D)$ and $f(b) \in Im(\partial_{n+1}^D) \in H_n(D)$. $\square$

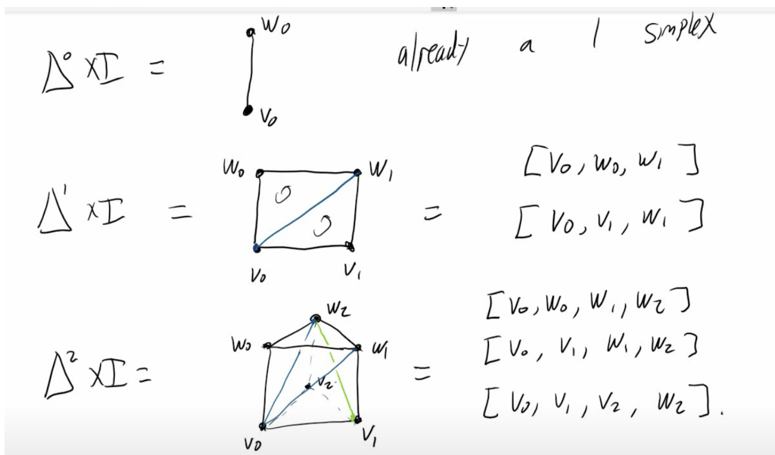
Lecture 17.

Review

A map $f : X \to Y$ induces a map $f_\# : C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$.
This map is a chain map since $f_\# \circ \partial = \partial \circ f_\#$. So we get a map $f_* : H_n(X) \to H_n(Y)$.
We can check $(f \circ g)_* = f_* \circ g_*$ and $id_* = id$ where the former is map induced by $id : X \to X$ and the latter is identity on $H_n(X)$.

Chain homotopies and homotopies

Recall that a homotopy between $f, g : X \to Y$ is a map $F : I \times X \to Y$ with $F\mid_{X\times\{0\}} = f$ and $F\mid_{X\times\{1\}} = g$.
Given a simplex Δ^n in X, a homotopy has a natural $\Delta^n \times I$ sitting inside of it.
So we need to understand $\Delta^n \times I$ in terms of simplices.
Some low dimension examples include $\Delta^0 \times I$ which is already a 1 simple. $\Delta^1 \times I$



In general, $\Delta^n \times I$ can be broken up into $n+1$ of the $n+1$ simplices, given by

$$\Delta^n \times I = \bigcup_i [v_0, \dots, v_i, w_i, \dots, w_n]$$

If $\sigma : \Delta^n \to X$ is a map then we get a map

$$\sigma \times id : \sigma \times I \to X \times I$$

given by $\sigma \times id(x, t) = (\sigma(x), t)$.

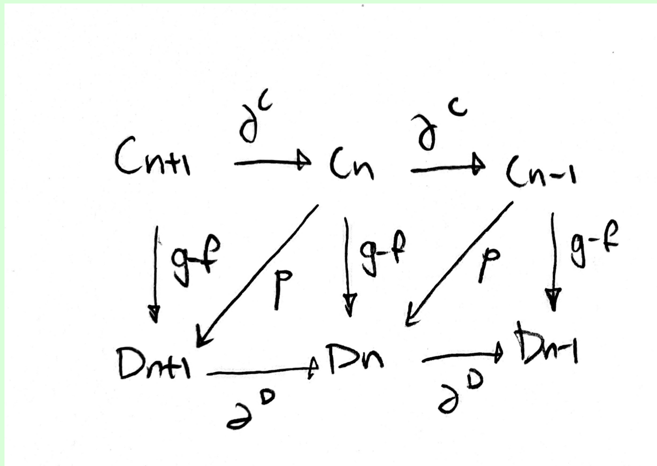
Definition: Given a homotopy $F : X \times I \to Y$, we define the prism operator to be the map $p : C_n(X) \to C_n(Y)$ given on basis elements as (whereas σ is an n-simplex)

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times id) \mid [v_0, \dots, v_i, w_i, \dots, w_n]$$

This means whenever we have the domain $\Delta^n \times I$, we can split it up into $n+1$ of $n+1$ simplices, and have a map that also maps simplices to the map $X \times I$. This means we can break $\Delta^n \times I$ into simplices as well.

Definition: Let (C_n, ∂^c) and (D_n, ∂^D) be chain complexes and let $f, g : C_n \to D_n$ be chain maps. Then a chain homotopy equivalence is a map $p : C_n \to D_{n+1}$ such that

$$P \circ \partial^C + \partial^D \circ P = g - f$$

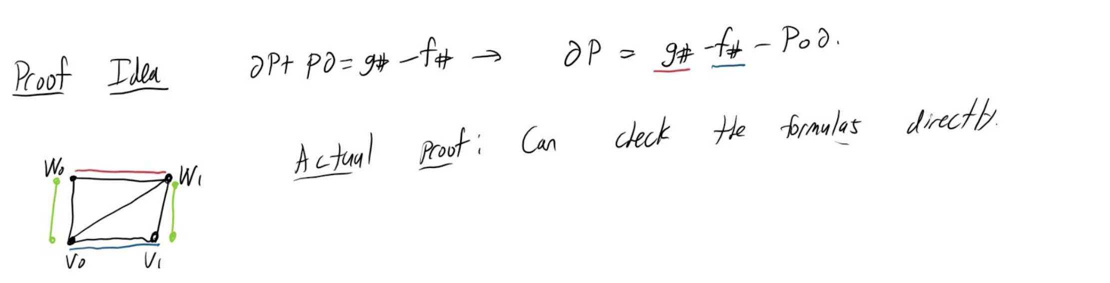


Note the above is not a commutative diagram!

Also the illustration above, the composition of map is written in wrong order.

Proposition 0.33:
If $F : X \times I \to Y$ is a homotopy between maps f and g then the prism operator $p : C_n(X) \to C_{n+1}(Y)$ is a chain homotopy between the maps $f_\#$ and $g_\#$. (that is $\partial p + p\partial = g_\# - f_\#$.)

Proof: Proof idea is that:
$\partial P + P\partial = g_\# - f_\#$, we can rephrase this as $\partial P = g_\# - f_\# - p \circ \partial$.
Remember the prism operator, takes $X \times I$, and cut it up to pieces, and map over to homotopy map. Note the boundary of p is broken up in this way. The actual proof can be check by formulas directly.



□

Note that $g_\#$ is the top and $f_\#$ the bottom, and $p \circ \partial$ is the sides. So ∂P is the entire boundary of the square.

Lemma 0.34:
If $f, g : (C_n, \partial^c) \to (D_n, \partial^D)$ are chain maps, which are chain homotopic, then f_* and $g_* : H_n(C) \to H_n(D)$ are the same map.

Proof:
If f, g are Chain homotopies, then $\exists P : C_n \to D_{n+1}$ so that $\partial P + P\partial = g_\# - f_\#$. If $\alpha \in C_n$ is a cycle, then $\partial P(C) + P\partial(C) = g_\#(C) - f_\#(C) \implies \partial P(C) = g_\#(C) - f_\#(C)$.
So $g_\#(C) - f_\#(C)$ is a boundary, so $g_*(C) - f_*(C) = 0 \in H_n(D)$ implies $g_*(C) = f_*(C)$.
Is the α here supposed to be a C instead? □

Theorem 0.35:
If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.
This shows that homology is a homotopy equivalence invariant of a space.

Proof: Let $h : Y \to X$ be the inverse if f so that $f \circ h \cong id_X$, $h \circ f \cong id_Y$. Then $(f \circ h)_* = f_* \circ h_*$ also $(f \circ h)_* = id$ so $f_* \circ h_* = id_{H_n(Y)}$. Similarly
$h_* \circ f_* = id_{H_n(X)}$ so f_* and g_* are inverse group isomorphisms. □

Here is a better explanation of chain homotopy:
`https://ncatlab.org/nlab/show/chain+homotopy`

Lecture 18. Free abelian groups

Definition 0.39: The free abelian group of rank n is the group $\mathbb{Z}^n$.

Lemma 0.36:
If B is a subgroup of $\mathbb{Z}^n$ then $B \cong \mathbb{Z}^m$ for $m \leq n$.

Proof: Proof is too long so omitted. □

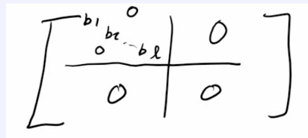
Homomorphisms on Free abelian groups

Free abelian groups are very similar to vector spaces. If G and G' are free abelian groups generated by $g_1, g_2, \dots g_n$ and $g'_1, g'_2 \dots g'_m$ respectively and $f : G \to G'$ is some homomorphism then we can represent f by a matrix.
If $f(g_i) = \sum_{i=1}^{m} a_{ij} g'_j$ then the matrix $A = (a_{ij})$ is such that

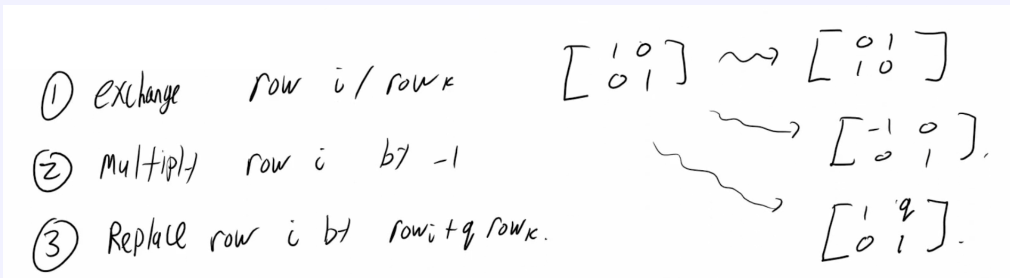
$$A \cdot g = g' \iff f(g) = g'$$

For example we can repressent a map $f : \mathbb{Z}^3 \to \mathbb{Z}^2$ by a 2×3 matrix.
Our goal is to find a nice form for homomorphisms which is amenable to homology calculations. This is called the Smith normal form.

Theorem 0.37:
Let G and G' be free abelian groups of rank n and m respectively. Then there are bases for G and G' such that the matrix representing an arbitrary form $f : G \to G'$ looks like:



with $b_1 \mid b_2 \mid b_3 \ldots b_{l-1} \mid b_l$
We will given an algorithm for this.
Given an arbitrary bases for G and G', we can modify the matrix by



This new matrix can be viewed as the same map on difference basis:

1. exchanges g_i and g_k
2. replaces g_i by $-g_i$
3. replaces g_i by $g_i - qg_k$

Similarly there are column operations which do thhe same thing to g_i'.

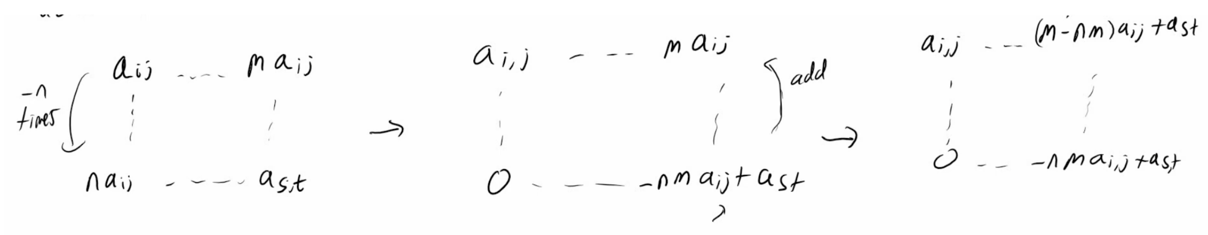
Using these operations we will get to smith normal form.
Given a matrix $A = (a_{ij})$, not all zero. Let $min(A) = min_{i,j}|a_{i,j}|$, We will proceed in two steps.

1. reduce $min(A)$
2. reduce matrix size

1. Claim: If $min(A) = a_{ij}$ does not divide some entry in its row or column then we can reduce the size of $min(A)$.

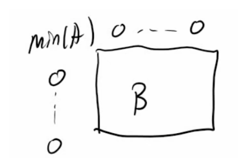
   To see this, if $min(A) \nmid akj$. Divide with remainder to get $a_{kj} = min(A) \cdot q + r$ with $|r| < |min(A)|$. Replace row k by row k-q row ji. Then a_{kj} becomes r, which replaces $min(A)$ by a new $a_{k,j}$ and go back to step 1.

   Now, suppose that $min(A) = a_{ij}$ divides all entries in its row/col but does not divide some other $a_{s,t}$. Consider the following operations:



Now a_{ij} does not divide the shown entry in its row. Then we can do the same thing as before to reduce $min(A)$.

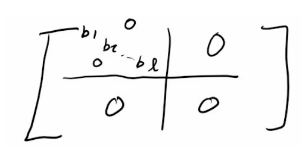
2. St the start we have a matrix where $min(A)$ divides every other entry in the matrix. Move $min(A)$ to top left and perform row/col operations to make all other elements in the leftmost column and topmost row to be 0. Repeat the algorithm in the submatrix B.



Stop when B is the 0 matrix or empty. At the end we have a matrix in the smith normal form. Since at the end of each step 2 we have a matrix where b_i divides all elements in the matrix, and row/col operations don't change this. So $b_1 \mid b_2 \mid \dots b_l$.

Smith normal form and homology

If $\partial_i : \mathbb{Z}^m \to \mathbb{Z}^n$ is given by the matrix



then

1. $Im(\partial_i) = b_1\mathbb{Z} \oplus b_2\mathbb{Z} \oplus \dots \oplus b_k\mathbb{Z}$ and
2. $Ker(\partial_i) = \{0\} \oplus \{0\} \oplus \dots \oplus \{0\} \oplus \mathbb{Z}^{m-k}$

Can we make ∂_{i-1} compatible at the same time?

Corollary 11: Let (C_n, ∂) be a chain complex of a finitely generated free abelian groups. Then for each i, there are subgroups u_i, v_i, w_i such that

$$c_i = u_i \oplus v_i \oplus w_i$$

where $\partial_i(u_i) \subset w_{i-1}, \partial(v_i) = 0, \partial(w_i) = 0$.
v_i is the free part, and w_i is the torsion part of the group.
$\mathbb{Z}^n$ is the free part, and $\mathbb{Z}/b_i\mathbb{Z}$ is the torsion part.
When diagonalize, all the nonzero parts u_i, maps to w_i, all in bottom i are the v_i and w_i. Look at smith normal form one level up, the partial of the new ones land inside w_i.

Proof: Let $w_i = \{w_i \in c_i \mid nw_i \in Im(\partial_{i+1}), n \in \mathbb{Z} \mid \{0\}\}$.
Then $n\partial_i(w_i) = \partial_i(nw_i) = 0 \implies \partial(w_i) = 0 \implies w_i \subset ker(\partial_i)$.
Rest of the kernel is v_i, rest of the group is u_i.
The upshot is, with basis as above

$$H_n(X) = ker(\partial_i)/Im(\partial_{i+1}) = v_i \oplus w_i/Im(\partial_{i+1}) = v_i \oplus (w_i/Im(\partial_{i+1}))$$

if ∂_{i+1} is written in the smith normal form.
Then

$$w_i/Im(\partial_{i+1}) = \mathbb{Z}/b_1^i\mathbb{Z} \oplus \mathbb{Z}/b_2^i\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_l^i\mathbb{Z}$$

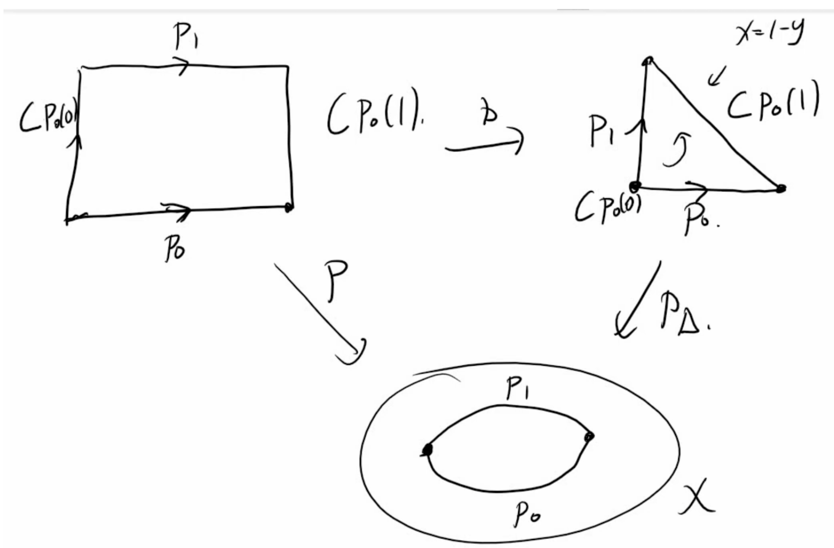
$\square$

Lecture 19 Homology and the fundamental group

Please resolve to J.Lee textbook "intro to top. mani. PAGE 351!!!!!!
First homology is completely determined by the fundamental group of the space, which is called the abelianization of the space.

Abelianization

If G is a group, its commutator subgroup $[G,G]$ is normally generated by elements of the form $[x,y] = xyx^{-1}y^{-1}$. Note that if $[x,y]=1$ then $xy=yx$. Then, $G/[G,G]$ is abelian and is called the abelianization of G, denoted by $Ab(G)$. Basically, abelianization measures how the group fails to be abelian.
Example:
Let $G = D_n = \langle r,s \mid r^n = s^2 = 1, rs = sr^{n-1}\rangle$. In $Ab(G)$, we quotient by $[r,s] = srs^{-1}r^{-1} = srsr^{n-1}$ $r^{n-1}s^2r^{n-1} = r^{2n-2} = r^{-2}$.
We need to quotient out by $\langle r^2\rangle$. This is normal so $D_n/\langle r^2\rangle$ is the abelianization and it is generated by $\overline{r}$ (image of r) and $\overline{s}$ (image of s).
If n is odd, then $r^n = 1 \implies r^{2k+1} = 1 \implies r^{2k}r = 1$ so $\overline{r}^{2k}\overline{r} - 1$. Still have $\overline{s}^2 = 1$. So $Ab(D_{2k+1}) = \mathbb{Z}/2\mathbb{Z}$.
If n is even, then $r^n = 1 \implies r^{2k} = 1 \implies \overline{r}^{2k} = 1$, this is known since $\overline{r}^2 = 1$ still $\overline{s}^2 = 1$. Also $rs = sr$ so $Ab(D_{2k}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Consider this:
If H is abelian and $f: G \to H$ is a homomorphism then $f([x,y]) = 0$, $\forall x,y \in G$. Then f factors through the abelianization. So the following diagram commutes.

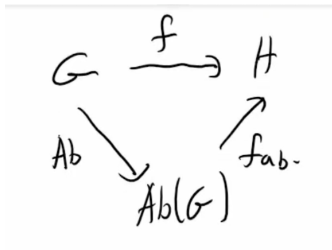


$\Pi_1(X)$ and $H_1(X)$

If $l: I \to X$ is a loop, then $[l]_\pi \in \pi_1(X)$. We can also regard I as an 1−simplex in Δ^1 with $l(0) = l(1)$. So l is a cycle $[l]_H \in H_1(X)$.
We will get a map $\gamma : \pi_1(X) \to H_1(X)$ given by $\gamma([l]_\pi) = [l]_H$.
Since $H_1(X)$ is abelian, this map will factor through $\gamma_{ab} = Ab(\pi_1(X)) \to H_1(X)$. We will see that γ_{ab} is also an isomorphism.

Lemma:
γ is well defined. More exactly, if $p_0, p_1 : I \to X$ are homotopy of paths, then $p_0 - p_1$ is a boundary in $C_1(X)$.

Proof: Let $p : I \times I \to X$ be the homotopy between p_0 and p_1. Consider the map $b : I \times I \to \Delta^2$ given by $b(x, y) = (x - xy, xy)$.



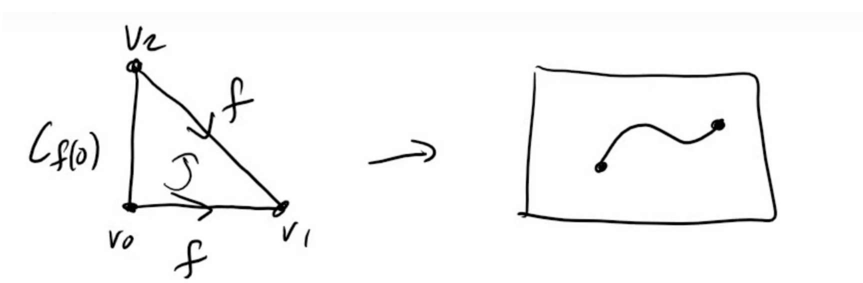
- If $x = 1$ then $b(x, y) = (1 - y, y)$
- If $x = 0$ then $b(x, y) = (0, 0)$
- If $y = 1$ then $b(x, y) = (0, x)$
- If $y = 0$ then $b(x, y) = (x, 0)$

Since b identifies points if and only if they are identified under p, so b factors to a map $p_\Delta : \sigma^2 \to X$.
$\partial p_\Delta = [p_0]_H + C_{p_0}(1) - [p_1]_H$. But $C_{p_0}(1)$ is ∂ of constant simple $\sigma_c : \delta^2 \to X$, $a \mapsto p_0(1)$, so $\partial(p_\Delta - \sigma_c) = [p_0]_H - [p_1]_H$.
So $[p_0]_H - [p_1]_H = 0 \in H_1$.

$\square$

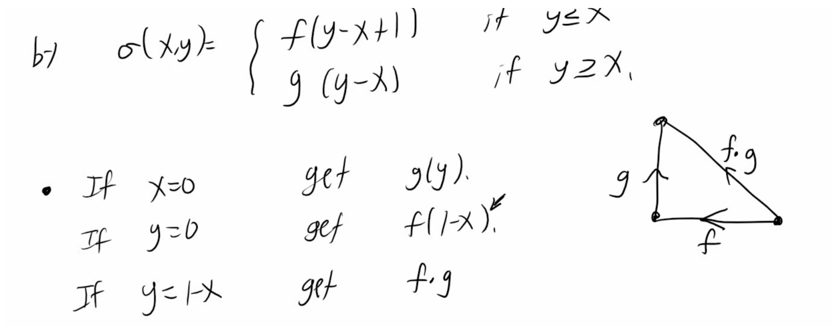
Lemma 0.38:
γ is a homomorphism

Proof: We start with $[\overline{f}]_H = -[f]_H$. To see this, define a singular 2 simplex $\sigma(x, y) = f(x)$.



$\partial\sigma = f + \overline{f} - C_{f(0)} \implies \partial(\sigma - \sigma_c) = \overline{f} + f$ so $f + \overline{f} = 0$ in $H_1(X)$. So $f = -\overline{f} \in H_1(X)$.

If f and g are paths with $f(1) = g(0)$ then we will show that $\gamma[f \cdot g] = [f]_H + [g]_H$.
Given such paths, define a 2 simplex $\gamma_p : \Delta^2 \to X$ by the following:

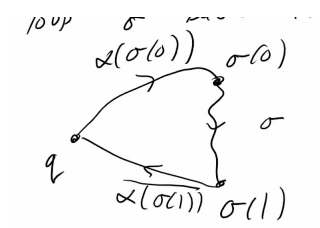


Check boundary and get that $[f \cdot g]_H = [f]_H + [g]_H$.

□

Lemma 0.39:
γ is surjective.

Proof: Fix a basepoint q and for all $x \in X$, let $\alpha(x)$ be a chosen path from q to x which $\alpha(q)$ be the constant path.
Since a point is a 0chain and a path is a 1 chain, this assignment extends to a homomorphism $\alpha : C_0(X) \to C_1(X)$.
For any path σ in X, define a loop $\tilde{\sigma}$ based at q by $\tilde{\sigma} = \alpha(\sigma(0)) \cdot \sigma \cdot \alpha\overline{\sigma(1)}$.



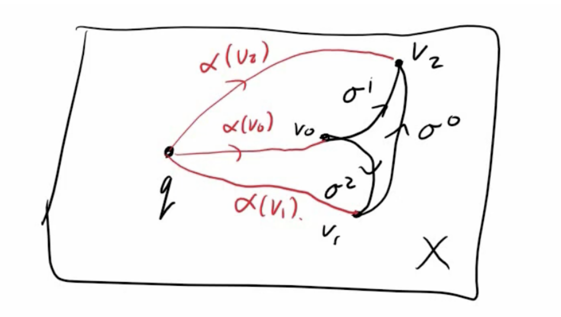
Now, $\gamma([\tilde{\sigma}]_\pi) = [\alpha(\sigma(0)) \cdot \sigma \cdot \alpha\overline{(\sigma(1))}]_H = [\alpha(\sigma(0))]_H + [\sigma]_H - [\alpha(\sigma(1))]_H = [\sigma]_H - [\alpha(\partial\sigma)]_H$.
Now suppose that $c = \sigma_{i=1}^m n_i \sigma_i$ is an arbitrary 1 chain.
Let f be the loop $(\tilde{\sigma_1})^{n_1} \cdot (\tilde{\sigma_2})^{n_2} \cdot \ldots \cdot (\tilde{\sigma_m})^{n_m}$. Then

$$\gamma([f]_\pi) = \sum_{i=1}^m n_i [\sigma_i]_H - [\alpha(\partial\sigma_i)]_H = [C]_H - [\alpha(\partial c)]$$

If c is a cycle then $\gamma([f]_\pi] = c$ so γ is surjective on homomlogy. □

> **Lemma 0.40:**
> $ker(\gamma) = [\pi_1(X), \pi_1(X)]$.

Proof: Let $A = Ab(\pi_1(X))$. For any loop f, let $[f]_A$ be image of f under the quotient.
If σ is a 1−simplex. Let $\beta(\sigma) = [\tilde{\sigma}]_A$. Since $C_1(X)$ free abelian and A is abelian, β extends uniquely to a hom $\beta : C_1(X) \to A$. We will show that β takes boundaries to $1 \in A$.
Let $\sigma : \Delta^2 \to X$ be a 2 simplex in the image. We have the following pictures:



$\partial\sigma = \sigma^0 - \sigma^1 + \sigma^2$. Note that $\sigma^0, \overline{\sigma^1}$ and σ^2 is homotopic to a constant loop.
Now, all in the group A, we have

Now

$$\beta(\partial\sigma) = \tilde{\sigma}^0 \cdot \overline{\tilde{\sigma}^1} \cdot \tilde{\sigma}^2 = \overbrace{\alpha(v_1)\cdot\sigma^0\cdot\overline{\alpha(v_2)}}^{\tilde{\sigma}(0)} \cdot \alpha(v_2)\,\overline{\sigma^1}\cdot\overline{\alpha(v_0)}\cdot \alpha(v_0)\cdot\sigma^2\cdot\overline{\alpha(v_1)}.$$

Δ is null hom

$$\cancel{\alpha(v_1)}\cdot\cancel{\sigma^0}\cdot\cancel{\overline{\alpha(v_2)}}\cdot\cancel{\alpha(v_2)}\,\cancel{\overline{\sigma^1}}\cdot\cancel{\overline{\alpha(v_0)}}\cdot\cancel{\alpha(v_0)}\cdot\cancel{\sigma^2}\cdot\cancel{\overline{\alpha(v_1)}}.$$

So $Im(\partial_2) \subset Ker(\beta)$.
Now suppose f is a loop based at q with $f_\pi \in ker\gamma$. Then $[f]_H = 0$ so that f is a boundary. Then since f is a loop at q, $\beta(f) = [\tilde{f}]_A = [f]_A$ but since β of a boundary is 0, $[f]_A = 1$. $\implies$ f is in $[\pi_1(X), \pi_1(X)]$.
Since $H_1(X)$ is abelian, $ker(\gamma)$ must contain $[\pi_1(X), \pi_1(X)]$. So this is exactly the kernel.
Summary of the theorem is, if X is a path connected then $\gamma : \pi_1(X) \to H_1(X)$ given $[f]_\pi \to [f]_H$ is a surjective group hom with $ker[\pi_1(X), \pi_1(X)]$. So $Ab(\pi_1(X)) = H_1(X)$.

□

Some quick applications

- $H_1(S^n) = 0$ for $n > 1$
- $H_1(\mathbb{R}P^2) = Ab(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$
- $\pi_1(S^1 \vee S^1) = Ab(\langle a, b \mid \rangle) = Ab(\langle a, b \mid ab = ba\rangle) = \mathbb{Z} \times \mathbb{Z}$.

Lecture 20. Exact sequences

Refer to J Lee Intro to Topological Manifolds page 356

Exact sequences

<u>Goal:</u> If $X = U \cup V$ with U, V open then we have inclusions

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X. \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

For all n, we want to define a map $\partial_* : H_n(X) \to H_{n-1}(U \cap V)$ such that the following sequence is exact:

$$\dots \xrightarrow{\partial_*} H_n(U \cap V) \xrightarrow{i_* \oplus j_*} H_n(U) \oplus H_n(V) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(U \cap V) \dots$$

Note that this is called **connecting homomorphism** which connects homology from an upper level into a lower level.

Definition 0.40 (Exact): A sequence of homomorphisms

$$\xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots$$

is said to be exact if for all n,

$$Ker(\alpha_n) = Im(\alpha_{n+1})$$

These are chain complexes with 0 homology at every level.

Many familiar relations can be expressed as exact sequences as follows:

1. $0 \to A \xrightarrow{\alpha} B$ exact means α is injective
2. $A \xrightarrow{\alpha} B \to 0$ exact means α is surjective
3. $0 \to A \xrightarrow{\alpha} B \to 0$ exact means α is an isomorphism
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact means α is injective, β is surjective with $Ker(\beta) = Im(\alpha)$. So $C = B/Im(\alpha) = B/A$ if we think of α as inclusion of A as subgroup of B.

An exact sequence like the one in 4 is called a SES, short exact sequence.
Recall that a chain map $F : (C_*, \partial^C) \to (D_*, \partial^D)$ is a map with $F(C_n) \subset D_n$ and $\partial^D \circ F = F \circ \partial^C$.
If C_*, D_*, E_* are chain complexes of abelian groups, then a sequence of chain maps
$C_* \xrightarrow{F} D_* \xrightarrow{G} E_*$ is exact if each of the sequences
$C_p \xrightarrow{F} D_p \xrightarrow{G} E_p$ is exact.
<u>Notation:</u> if $c \in C_p$ and $\partial_c = 0$, let $[c] \in H_p(C)$ be the homology class.

Lemma 0.41 (The zig-zag lemma):
Let $C_* \xrightarrow{F} D_* \xrightarrow{G} E_*$ be a SES of chain maps. Then for each p, there is a "connecting homomorphism " $\partial_* : H_p(E_*) \to H_{p-1}(C_*)$ such that the following sequence is exact:

$$\ldots \xrightarrow{\partial_*} H_p(C_*) \xrightarrow{F_*} H_p(D_*) \xrightarrow{G_*} H_p(E_*) \xrightarrow{\partial_*} H_{p-1}(C_*) \xrightarrow{F_*} H_{p-1}(D_*) \xrightarrow{G_*} \ldots$$

(the long exact sequence)
The word long exact sequence comes from when you give a short exact sequences, it returns u a long exact sequence.

Proof: Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "F"] \arrow[d, "\partial"] & D_{p+1} \arrow[r, "G"] \arrow[d, "\partial"] & E_{p+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_p \arrow[r, "F"] \arrow[d, "\partial"] & D_p \arrow[r, "G"] \arrow[d, "\partial"] & E_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "F"] \arrow[d, "\partial"] & D_{p-1} \arrow[r, "G"] \arrow[d, "\partial"] & E_{p-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "F"] & D_{p-2} \arrow[r, "G"] & E_{p-2} \arrow[r] & 0.
\end{tikzcd}$$

The proof will be using diagram chase. I will omit the details. The goal would be to get from $E_p(E)$ down to $E_{C_{p-1}}$ somehow. $\square$

Lemma 0.42 (Five lemma):
Let A_i, B_i be abelian groups such that the diagram commutes and has exact rows.

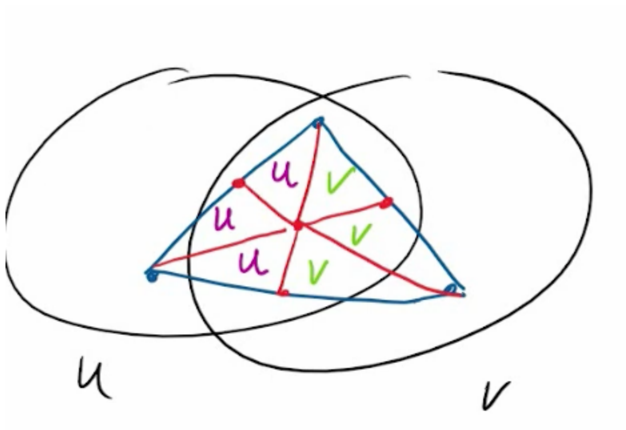
$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

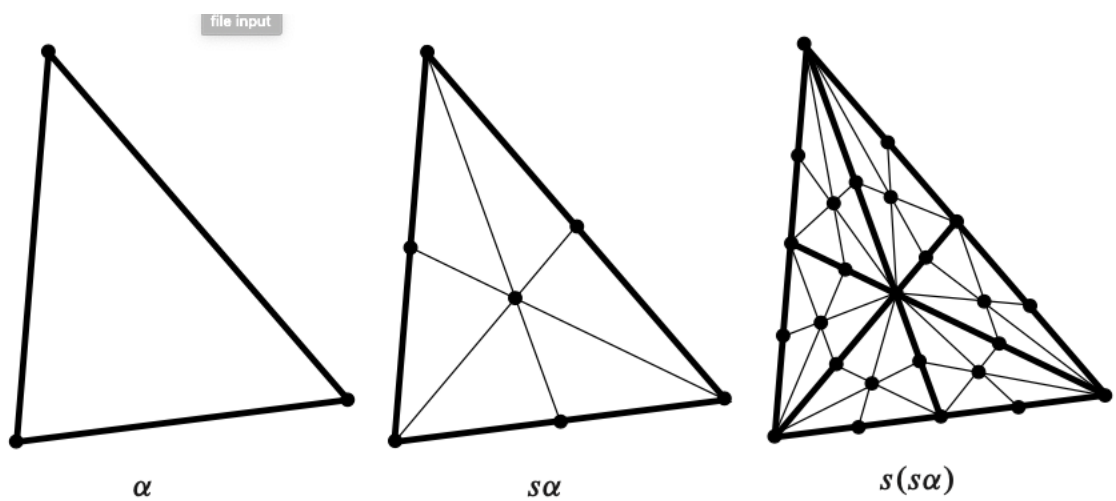
Proof: Proof is by diagram chase again. $\square$

Lecture 21. The Mayer-Vietoris Sequences

<u>Setup</u>
Let $X = U \cup V$ for U, V open. We want to compute the homology of X in terms of U and V. This can be seen as an analogue of VKT of $\pi_1(X)$. There, we expressed loops in X as product of loops in U and V. In here, we need to do something similar.
First hurdle: A chain $\sigma : \Delta^n \to X$ may not land completely in either U or V.



The solution is to divide up chains into little pieces, each of which fits in U or V.
The division of a simplex like this is called barycentric subdivision.



(Lebesgue number lemma).

U-small homology

Let $X = \bigcup_\alpha U_\alpha$ be an open cover. Define $C_n^U(X)$ be the free abelian group generated by simplices $\sigma : \Delta^n \to X$ so that $\sigma(\Delta^i) \subset U_i$ for some i. Note that if $\sigma_i \in C_n^U(X)$ then $\partial\sigma_i \in C_{n-1}^U$.
So we get a homology theory, $U-$small homology. We denote it by $H_n^U(X)$.
We want to get a map $S_u : C_n(X) \to C_n^U(X)$.
We define this on a basis element: $\delta : \Delta^n \to X$.

1. Divide all 1 simplices in Δ^n in half by placing a vertex at the center of each edge
2. Put a point in the middle of each 2 simplex face and draw a line between each vertex and that new vertex in the middle.
3. Put a point in the middle of each 3 simplex face and draw a line from each vertex to the new point.
4. Continue inductively up to N simplices
5. In the end, we define $S(\sigma^n)$ to the signed sum of resultnig simplices.

We can extend this to a map $S : C_n(X) \to C_n(X)$ by giving the barycentric divisions.
Define $S^m(\sigma) = S(S^{m-1}(\sigma))$. If $X = \bigcup_\alpha U_\alpha$ is an open cover and $\sigma : \Delta^n \to X$ is a map, then for some m, each simplex of $S^m(\sigma)$ lands in a single U_i. The key word is lebesgue number lemma. N simplex is complex and U_α is open, but keep doing it will eventually have the simplices sitting inside U_α.
Define $S_u(\sigma)$ be the minimal m, $s^m(\sigma)$ such that $S_m(\sigma)$ fits in U_i.
We defined this on generators, we can now extend this into a map

$$S_U : C_n(X) \to C_n^u(X)$$

Lemma 0.43:

1. If $f : X \to Y$ and U_α, V_β are open covers of X and Y, then $S_V \circ f_\# = f_\# \circ S_U$
2. $\partial \circ s = s \circ \partial$ so s is a chain map.

Proposition 0.44:
$S^m : C_n(X) \to C_n(X)$ is chain homotopic to id

Proposition 0.45:
$H_n^u(X)$ is isomorphic to $H_n(X)$.

Mayer Vietoris Theorem

If $X = A \cup B$ with A, B open then we have inclusions (replace U, V with A, B)

$$\begin{tikzcd}
& U \arrow[dr, "k"] & \\
U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X. \\
& V \arrow[ur, "l"'] &
\end{tikzcd}$$

Then for each p there are connecting homomorphisms $\partial_* : H_n(X) \to H_{n-1}(A \cap B)$ so that the following sequence is exact

$$\dots \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{i_* \oplus j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \dots$$

Note that this is called **connecting homomorphism** which connects homology from an upper level into a lower level.

Proof: Let $U = A \cup B$. Consider the maps

$$C_n(A \cup B) \xrightarrow{i_\# \oplus j_\#} C_n(A) \oplus C_n(B) \xrightarrow{k_\# - l_\#} C_n^U(X)$$

We claim this is a SESS.
Since $i_\#, j_\#$ are just maps which takes a simplex in $A \cup B$ and regard it as a simplex in A or in B, this map is injective.
If $C \in C_n(A \cap B)$, then $(k_\# - l_\#)((i_\# \oplus j_\#)(c)) = 0$. That is, this is similar to include it in i, and include it in j, and identifying them as the same thing, so we subtract it off.
In fact this is the entire kernel so it is exact at $C_n(A) \oplus C_n(B)$.
Next check $k_\# - l_\#$ is surjective by the definition of $C_n^U(X)$. (Since C_n^U simplices are generated by elements either entirely in A Or B, those are contained, and it will hit over the element.)
This is a SES of chain complexes. Indeed. Now, we get a long exact sequence of homologies.
It looks like

$$\xrightarrow{\partial_*} H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n^U(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B)$$

But by prop $H_n^U \cong H_n$ so replacing it with H_n above gives us desired result.

$\square$

Applications

<u>$H_i(S^n)$</u>
By a similar decomp to VKT, we can find $H_i(S^n)$.
Let A =northern hemi plus a bit, B =southern hemi plus a bit.
So $A, B \cong^{h.e.q} D^n$ and $A \cap B \cong^{h.e.q} S^{n-1}$. By MVS, we get a LES

$$H_{n+1}(S^{n-1})^{=A\cap B} \to H_{n+1}(D^n)^{=A} \oplus H_{n+1}(D^n)^{=B} \to H_{n+1}(S^{n+1})^{=X} \to H_n(S^{n-1})$$

$$\to H_n(D^n) \oplus H_n(D^n) \to H_n(S^n) \to H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \oplus H_{n-1}(D^n)$$

Recall we showed that $H_i(S^1) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{o.w.} \end{cases}$

We will show by induction that $H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{o.w.} \end{cases}$

Suppose it was true for S^{n-1}, then we can substitute the above sequence.

$$H_{n+1}(S^{n-1})^0 \to H_{n+1}(D^n)^0 \oplus H_{n+1}(D^n)^0 \to H_{n+1}(S^{n+1})^? \to H_n(S^{n-1})^0$$

$$\to H_n(D^n)^0 \oplus H_n(D^n)^0 \to H_n(S^n)^? \to H_{n-1}(S^{n-1})^{\mathbb{Z}} \to H_{n-1}(D^n)^0 \oplus H_{n-1}(D^n)^0$$

- $0 \to H_{n+1}(S^n) \to 0 \to 0$ so $H_{n+1}(S^n) = 0$ as sandwiched between two zeroes, get isomorphic, so it is zero. Similarly, $H_{n+k}(S^n) = 0$ for $k > 0$.
- $0 \to H_n(S^n) \to \mathbb{Z} \to 0 \implies H_n(S^n) = \mathbb{Z}$
- for $0 < k < n$, $0 \to H_k(S^n) \to 0 \to 0$ implies $H_k(S^n) = 0$ for $0 < k < n$.

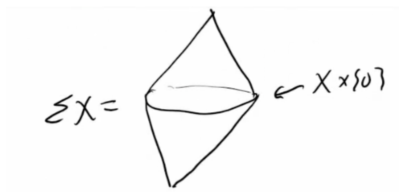
Also since S^n is connected $H_0(S^n) = \mathbb{Z}$. In summary

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{o.w.} \end{cases}$$

Corollary 12: $\mathbb{R}^n \neq^{homeo} \mathbb{R}^m$ for $n \neq m$. Since $R^n \setminus \{pt\} \cong S^{n-1}$, $R^m \setminus \{pt\} \cong S^{m-1}$, but homology groups are homotopic invariance, so they are certainly not homeo.

Suspension

Definition 0.41: If X is a top space, then we define the suspension of X, denoted ΣX, to be $X \times I/\sim$ where $\forall x, y \in X, (x, 0) \cong (y, 0)$ and $(x, 1) \cong (y, 1)$. If $f : X \to Y$ is a map, we denote by Σf the map $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f(a, t) = (f(a), t)$.



For example $\Sigma S^1 = S^2$ and $\Sigma S^n = S^{n+1}$.

Proposition 0.46:
For $n \geq 1$, $H_{n+1}(\Sigma X) = H_n(X)$.

Proof: Take a similar decomp of Σx to S^n. Let $A = X \times [0, 3/4]$ and $B = X \times [1/4, 1]$ both A and B are contractible, we have $A \cap B \cong X$. So by MVS, we have LES:

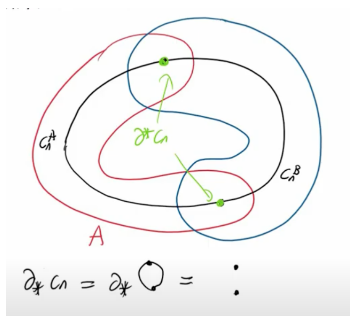
$$H_n(X) \to 0^{=H_n(A) \oplus H_n(B)} \to H_n(\Sigma x) \xrightarrow{\partial_*} H_{n-1}(X) \to 0^{=H_{n-1}(A) \oplus H_{n-1}(B)}$$

So we get two zeros, the ∂_* again is an isomorphism. $\square$

Lecture 22. Mayer Vietoris Applications

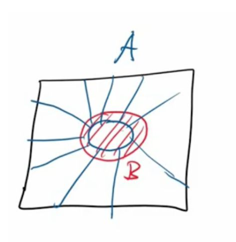
The connecting homomorphism

Note that i_*, j_*, k_*, l_* all come from inclusion. The map ∂_* was defined algebraically via the zigzag lemma. We want to define ∂_* geometrically. $\partial_* : H_n(X) \to H_{n-1}(A \cap B)$. Need to take an n cycle and return an $n-1$ cycle in $A \cap B$.
Suppose c_n is some n cycle in X. By barycentric subdivision, we can break up c_n into a sum of simplices, each of which is contained entirely in A or B.
The following is an example. Note that $A \cap B$ does not need to be connected.
we can see the following example:
The black circle is C_n, the n chain, and we broke it up to be in either entirely in A and B.
Call these C_n^A, C_n^B.
Then $\partial C_n = 0$ as it is a cycle, so $\partial(C_n^A) + \partial(C_n^B) = 0$ so $\partial(C_n^A) = -\partial(C_n^B)$. So we conclude that boundaries of these simplices must lie in $A \cap B$.
Define $\partial_* C_n = [\partial C_n^A] \in H_{n-1}(A \cap B)$.
But here, we made a choice to pick to choose ∂C_n^A instead of ∂C_n^B. These maps are negations of eachother. So while the map on the MVS level depends on which space A/B, the resulting groups do not depend, since $Im/Ker(f) = Im/ker(f)$.



Surfaces

Calculate homology groups of T^2.
Let $T^2 = A \cup B$ where $A = T^2 \setminus$ small disk and $B =$ slightly bigger disk



So $A =$wedge of two circles, $B =$point, and $A \cap B = S^1$.
Also we know the homology of those sets:

$$H_n(A) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}, H_n(B) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{else} \end{cases}, H_n(A \cap B) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

If $n \geq 3$, then MVS looks like:

$$0^{=H_n(A\cap B)} \to \underline{0^{=H_n(A)\oplus H_n(B)}} \to H_n(T^2) \to 0^{H_{n-1}(A\cap B)} \to \underline{0^{=H_{n-1}(A)\oplus H_{n-1}(B)}}$$

Looking at the two underlined terms, they are both zero, so the two groups in between are isomorphic. This means $H_n(T^2)$ is zero for $n \geq 3$.
If $n = 2$ then the MVS looks like:

$$0^{=H_2(A\cap B)} \to 0^{=H_2(A)\oplus H_2(B)} \to H_2(T^2) \xrightarrow{\partial_*} \mathbb{Z}^{H_1(A\cap B)} \xrightarrow{i_*\oplus j_*} \mathbb{Z}\oplus\mathbb{Z}^{=H_1(A)\oplus H_1(B)}$$

Since $H_2(A) \oplus H_2(B) = 0$, $H_2(T^2)$ injects into $\mathbb{Z}$.
A generator for $H_1(A\cap B)$ is a loop, j_* is clearly 0 since it lands in $H_1(B) = 0$. When including into A, it goes around $aba^{-1}b^{-1}$ inside homology of wedge of two circles in $\mathbb{Z}\oplus\mathbb{Z}$. So $i_*(1) = aba^{-1}b^{-1} \in H_1(\infty) = \mathbb{Z}\oplus\mathbb{Z}$. Since, direct sum is commutative, i_* is also the 0 map. So $ker\ i_* \oplus j_* = \mathbb{Z} \implies Im(\partial_*) = \mathbb{Z}$, so ∂_* is surjective. Since ∂_* is injective, it is bijective and we have $H_2(T^2) = \mathbb{Z}$.
At $n = 1$, the MVS look like:

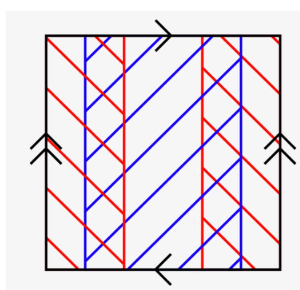
$$\mathbb{Z}^{=H_1(A\cap B)} \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z}^{=H_1(A)\oplus H_1(B)} \xrightarrow{g} H_1(T^2) \xrightarrow{h} \mathbb{Z}^{H_0(A\cap B)} \xrightarrow{i=i_*\oplus j_*} \mathbb{Z}\oplus\mathbb{Z}^{=H_0(A)\oplus H_0(B)}$$

(Wait, I think $H_1(B) = 0$?) We have seen $f = 0$ so $ker(g) = 0$ so $\mathbb{Z}\oplus\mathbb{Z}$ injects into $H_1(T^2)$. Now i takes $1 \mapsto 1\oplus 1$ so i is injective means $Im(h) = Ker(i) = 0$. Therefore since h maps everything to 0, $ker(h) = H_1(T^2)$ so $im(g) = H_1(T^2)$. So since g is both surjective and injective, it is bijective so we have $H_1(T^2) = \mathbb{Z}\oplus\mathbb{Z}$.
Also $H_0(T^2) = \mathbb{Z}$ as it is a path connected space.
In summary,

$$H_n(T^2) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$

Calculate the homology groups of the Klein bottle K

Let A/B be as in the picture. Then $A, B, A\cap B$ are all mobius bands.



Let A be the blue strip in the middle and B be the red regions.
Then

$$H_n(all) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

As before, $H_n(K) = 0$ for $n > 3$. At $n = 2$ we have

$$0^{=H_2(A)\oplus H_2(B)} \to H_2(K) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i} \mathbb{Z}\oplus\mathbb{Z}$$

So $H_2(K) = Im\partial_* = ker(i)$. i is given by including into A and into B. When you have a loop in intersection, i.e. going in between twice loops, when u include it into A and B, it goes to the boundary of the mobius band, which is equal to 2 in the $H_1(A), H_1(B)$. So $i_*(1) = j_*(1) = 2$. So $i_* \oplus j_*(1) = (2, 2)$. So i is injective. Since by exactness ∂_* is injective, and $0 = ker(i) = im(\partial_*)$ but ∂_* is injective, so $H_2(K)$ must only have the identity element. So $H_2(K) = 0$. Then, at $H_1(K)$ we have

$$\mathbb{Z}^{=H_1(A\cap B)} \to \mathbb{Z} \oplus \mathbb{Z}^{H_1(A)\oplus H_1(B)} \xrightarrow{f} H_1(K) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}.$$

Since $A \cap B$ is connected, subdividing any loop into smaller loops results in an even number of points in the same component. So $\partial_* = 0$ so f is surjective.
why is this 0? and why does it make it even number points?
please revisit this component!!
please revisit this component!!
please revisit this component!!
So

$$H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/ker(f) = \mathbb{Z} \oplus \mathbb{Z}/IM(g), \text{or } i_* \oplus j_* = \mathbb{Z} \oplus \mathbb{Z}/span(2, 2)$$

Now $\mathbb{Z} \oplus \mathbb{Z} = span((1, 0), (0, 1)) = span((1, 0), (1, 1)) = \mathbb{Z}(1, 0) \oplus \mathbb{Z}(1, 1)$.
So $\mathbb{Z} \oplus \mathbb{Z}/span(2, 2) = \mathbb{Z}(1, 0) \oplus \mathbb{Z}(1, 1)/span(2, 2) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.
In summary,

$$H_n = \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

Knot complements

Recall that if $K \subset S^3$ is a not then $\pi_1(S^3 \setminus K)$ is a powerful invariant of that knot. How does $H_1(S^3 \setminus K)$ work as a knot invariant?
Let us use the Meyer vietoris sequence to figure this out.
By the definition of knot, a knot can be “thickened up” to an embedding of $S^1 \times D^2$. Given a knot K, let $\nu(K)$ be a slight thickening of K. Also let $\nu^+(K)$ be a slightly larger thickening of K.
We define $A = \nu^+(K)$ and $B = S^3 \setminus \nu(K)$. Note that $S^3 = A \cup B$. Now, also $A \cap B = T^2 \times I \cong T^2$.
I don't see why this is is true. I only see how it is a flat annulus times I.
At $n = 1$, the MVS is

$$0^{=H_2(S^3)} \xrightarrow{\partial_*} \mathbb{Z} \oplus \mathbb{Z}^{H_1(A)\cap H_1(B)} \to \mathbb{Z}^{=H_1(A)} \oplus H_1(B) \to 0^{=H_1(S^3)}$$

So $H_1(B) = H_1(S^3 \setminus \nu(K)) = \mathbb{Z}$.
In conclusion $H_1(S^3 \setminus \nu(K))$ can not distinguish knots. This is part of a larger phoenomena.

Theorem 0.47:
If $h : S^k \to S^n$ is an embedding with $k < n$. Then

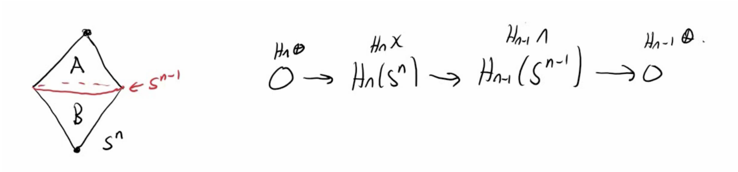
$$H_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1, 0 \\ 0 & \text{else} \end{cases}$$

Lecture 23. Degree theory for spheres

Recall

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases}$$

By MVS,



Recall
A map $f : X \to Y$ induces a map $f_* : H_k(X) \to H_k(Y)$.
Then a map $f : S^n \to S^n$ induces a map $f_* : H_n(S^n) \to H_n(S^n)$, which is $f_* : \mathbb{Z} \to \mathbb{Z}$. Such maps $f_* : \mathbb{Z} \to \mathbb{Z}$ are completely determined by $f_*(1)$.

Definition 0.42: $f : S^n \to S^n$ is a degree n map if $f_* : H_n(S^n) \to H_n(S^n)$ has $f_*(1) = n$.

Example
If $f : S^1 \to S^1$ is $f(z) = z^n$, then f has degree n.

Note that the maps sends the one generator of one simplex to the copy of three simplices in its image.

Some straightforward properties

1. $deg(id_{S^n}) = 1$ since it induces the identity map $\mathbb{Z} \to \mathbb{Z}$.
2. If $g \simeq g$ then $deg(f) = deg(g)$ since $f_* = g_*$.
3. If f is not surjective, then $deg(f) = 0$. Since $f \simeq$ constant map. That is, if it misses a point, then it would be homotopic equivalent to $\mathbb{R}^n$, which is contractible.
4. $deg(f \circ g) = \deg(f)\deg(g)$ since $(f \circ g)_* = f_* \circ g_*$.

Proposition 0.48 (Naturality of connecting homomorphism):
Suppose we have a commutative diagram (of chain complexes) of exact rows, given by

$$\begin{tikzcd} 0 \arrow[r] & C_* \arrow[r] \arrow[d, "\kappa"] & D_* \arrow[r] \arrow[d, "\delta"] & E_* \arrow[r] \arrow[d, "\epsilon"] & 0 \\ 0 \arrow[r] & C'_* \arrow[r] & D'_* \arrow[r] & E'_* \arrow[r] & 0 \end{tikzcd}$$

Then the following diagram commutes for each p. Note that this is the same setup as in zig zag lemma.

$$\begin{tikzcd} H_p(E_*) \arrow[r, "\partial_*"] \arrow[d, "\epsilon_*"] & H_{p-1}(C_*) \arrow[d, "\kappa_*"] \\ H_p(E'_*) \arrow[r, "\partial_*"] & H_{p-1}(C'_*) \end{tikzcd}$$

Degree of reflections

Let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$.
Let $R_i : S^n \to S^n$ be the map $(x_1, \ldots, x_i, \ldots, x_{n+1}) \to (x_1, \ldots, -x_i, \ldots, x_{n+1})$.
If S_{ij} is the map that swaps x_i, x_j then $R_j = S_{ij}R_iS_{ij}$.
So

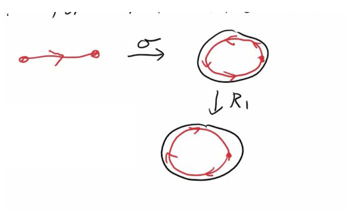
$$deg(R_j) = deg(S_{ij}R_iS_{ij}) = deg(S_{ij})deg(R_i)deg(S_{ij})$$
$$= deg(S_{ij})deg(S_{ij})deg(R_i) = deg(\underbrace{S_{ij}S_{ij}}_{id})deg(R_i) = deg(R_i).$$

So all $deg(R_i)$ have the same degree.

Lemma 0.49:

$$\deg(R_i) = -1$$

Proof:
We will induct on n.
<u>Base case</u>
For $n = 1$, take the generating 1-chain of H, given by



Compose where the 1 chain is mapped to by R_1, it goes to $-\sigma$. So $deg(R_1) = -1$ on S^1.

Inductive step

Recall in MVS, we found $H_n(S^n) \cong H_{n-1}(S^{n-1})$. (by ∂_*)

We seek to make this compatible with R_1. I.e. we want the following diagram to commute.

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\partial_*"] \arrow[d, "R_{1*}"] & H_{n-1}(S^{n-1}) \arrow[d, "R_{1*}"] \\
H_n(S^n) \arrow[r, "\partial_*"] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

Note that these R_{1*} technically are different maps, but the R_{1*} on bigger space does induce the same on the lower space.

Let $\mathcal{U} = U \cup V$ be the open cover of S^n given by $U =$ northern hemi + little, and $V =$ southern hemi + little.

Note that R_1 preserves U and V. Note that R_1 reflects across the vertical axis, and the nothern hemi, southern hemi are preserved by R_1.

Then R_1 induces chain maps:

$$\begin{tikzcd}
0 \arrow[r] & C_*(U \cap V) \arrow[r] \arrow[d, "R_{1\#}"] & C_*(U) \oplus C_*(V) \arrow[r] \arrow[d, "R_{1\#} \oplus R_{1\#}"] & C_*^{\mathcal{U}}(S^n) \arrow[r] \arrow[d, "R_{1\#}"] & 0 \\
0 \arrow[r] & C_*(U \cap V) \arrow[r] & C_*(U) \oplus C_*(V) \arrow[r] & C_*^{\mathcal{U}}(S^n) \arrow[r] & 0
\end{tikzcd}$$

By the naturality of ∂_*, we get a commutative square of maps on homology as follows: (Using the fact that $\mathcal{U}$ small homology is isomorphic to original homology.)

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\partial_*"] \arrow[d, "R_{1*}"] & H_{n-1}(S^{n-1}) \arrow[d, "R_{1*}"] \\
H_n(S^n) \arrow[r, "\partial_*"] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

Now, since R_{1*} on the right is the -1 map and the diagram is commutative, R_{1*} on the left is a degree -1 map.

$\square$

Recall that the antipodal map $A : S^n \to S^n$ is $A(\vec{x}) \to -\vec{x}$.

Corollary 13:

$$deg(A_{S^n}) = (-1)^{n+1}$$

Proposition 0.50:
$A : S^n \to S^n$ is homotopic to the identity $\iff$ n is odd.

Proof:
$\implies$: Contrapositive is : n even implies A not homomorphic to id. By previous corollary, if n is even, then $deg(A) = (-1)^{n+1} = -1$. Since $deg \neq 1$, and degree is invariant under homotopies, then the result follows.
$\impliedby$: Suppose $n = 2k - 1$ is odd. Then there is an explicit homomotopy (this is quite complicated, I omit the proof). We have $H(X, 0) = id$ and $H(x, 1) = A$. Each $H(x, t) : S^n \to S^n$ well defined.
A continuous vector field is a continuous map $V : S^n \to \mathbb{R}^{n+1}$ so that at every $x \in S^n, V(X)$ is tangent to S^n at X. Algebraically, $V(X) \cdot X = 0$. These play important role in physics and diff geo.
A vector field is said to be non-vanishing if $V(X) \neq 0$ at any x.
Consider the Hairy ball theorem. $\square$

Theorem 0.51 (Hairy Ball Theorem):
S^n admits a non-vanishing vector field $\iff$ n is odd.

Proof: $\impliedby$
If $n = 2k - 1$ is odd, then the vector field $V : S^{2k-1} \to \mathbb{R}^{2k}$ given by

$$V((X_1, \ldots, X_{2k})) \to (-x_2, x_1, -x_4, x_3 \ldots - x_{2k}, x_{2k-1})$$

is nonvanishing.
$\implies$
Suppose S^n admits a nonvanishing vector field. Let w be the new vector field. $w = \frac{v}{|v|}$. Note that $|w(x)| = 1$ and $x \cdot w(x) = x \cdot \frac{v}{|v|} = \frac{x \cdot v}{|v|} = 0$.
We have a explicit homotopy. $H(x, t)$. (Too lazy to write it down.) At $t = 0, H(x, t) = id, t = 1, H(x, t) = -\vec{x} = A$.
And the homotopy is 1 at any point. So $deg(A)$ is odd.

$\square$

Theorem 0.52:
If $f : S^n \to S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.

Proof: We can obtain a homootopy between f and A by

$$H(x,t) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

Recall $G \hookrightarrow x$ freely means there is a homomorphism $\hookrightarrow: G \to \text{homeo}(X)$ with $\forall x, g \cdot x \neq x$ unless $g = id$.$\square$

The following prop gives us a severe restriction on a group acting on a space.

Proposition 0.53:
If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group that can act on S^n.

Proof: Remark
We have already seen the converse, that $\mathbb{Z}/2\mathbb{Z}$ acts on S^n by antipodal map, so the quotient is $\mathbb{R}P^n$. So the universal cover of $\mathbb{R}P^n$ is S^n. It turns out that this is all that can happen.
If $f \in \text{Homeo}(S^n)$, then $\deg(f \circ f^{-1}) = \deg(f) \cdot \deg(f^{-1})$ so $\deg(f) = \pm 1$. So we get a composed homomorphism $d : G \to \{\pm 1\}$, defined by $G \to Homeo(X) \to \{\pm 1\}$.
If $G \hookrightarrow$ freely by the previous proposition, $\forall g$ with $g \neq id$, $d(g) = (-1)^{n+1} = -1$ since n even. Then d has trivial kernel. This is injective homomorphism. So $G \leq \mathbb{Z}/2\mathbb{Z}$. $\square$

The question of which groups act freely on S^n for n odd is much more subtle.

Lecture 24. Cellular Homology

Last time:
We showed $f : S^n \to S^n$ has degree n if $f_* : H_n(S^n) \to H_n(S^n)$ (or $\mathbb{Z} \to \mathbb{Z}$) has $f_*(1) = n$.

Interesting fact:
Two maps $f, g : S^n \to S^n$ are homotopic $\iff$ $\deg(f) = \deg(g)$.

Recall that a CW complex is built by attaching $n$cells (D^n) to the $n-1$ skeleton X^{n-1} by maps $f : S^{n-1} \to X^{n-1}$ where $S^{n-1} = \partial(D^n)$.

Goal: define a homology theory $H_*^{CW}(X)$ based on these attaching maps. For today, we let CW complexes have finitely many cells in each dimension.

Observation: If X is CW complex, then $X^n/X^{n-1} \simeq \vee_k S^n$ where k is the index of # of $n-$cells.
Then attaching an $n+1-$cell gives a map $f : S^n \to \vee_k S^n$ obtained by compositing the attaching map $f : S^n \to X^n$ with quotient map $X^n \to X^n/X^{n-1} \cong \vee_k S^n$.
So on homology we get a map $f_* : \mathbb{Z} \to \mathbb{Z}^n$. This sends $1 \mapsto (n_1, n_2, \ldots, n_k)$.
Can think of this as a multidegree. Further quotienting out by all but the ith sphere gives a map of degree n_i.
Let C_n^{CW} be the abelian group generated by the $n-$cells.
If e_i^n is an n-cell, it is attached by a map with multi-index $(n_i^1, n_i^2, \ldots, n_i^k)$. They each correspond to $e_1^{n-1}, e_2^{n-1}, \ldots, e_k^{n-1}$. We define $\partial_n : C_n^{CW} \to C_{n-1}^{CW}$ on each cell by $\partial e_i^n = \sum_{j=1}^k n_i^j e_j^{n-1}$. n_i is the number of times the cell is wrapped around the other boundaries. ∂_0 is the 0 map and ∂_1 is terminal vertex - initial.

Theorem 0.54:
$\partial^2 = 0$ so we get a homology theory H_*^{CW}. Moreover, if X is a CW complex, then

$$H_*^{CW}(X) = H_*^{\Delta}(X) = H_*(X)$$

0.5 Cellular homology examples

CW homology of Σ_g
Recall the CW decomposition of Σ_g is the $2n$ gon with boundary $[a_1 b_1] \ldots [a_y b_y]$.
So

- $C_0^{CW}(\Sigma_g) = \mathbb{Z}$
- $C_1^{CW}(\Sigma_g) = \mathbb{Z}^{2g}$
- $C_2^{CW}(\Sigma_g) = \mathbb{Z}$

Every $1-$cell has both ends attached to the same 0cell. So $\partial^1 = 0$. The $2-$cell is attached along $[a_1 b_1] \ldots [a_y b_y]$, we quotient out each generator, since each degree is 0, so they each cancel. So $\partial_2 = 0$.

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

So

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^{2g} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

It is faster than MVS, simplicial homology, it is almost impossible to work with singular homology.
We can try to work out $\mathbb{R}P^2$ or more general non-orientable surfaces.
Homology of T^3
We consider the CW decomposition of the T^3. The opposite sides are identified with each other. There is only 1 $0-$cell so $\partial_1 = 0$. There are 3 $2-$cells all attached like T^2. So $\partial_2 = 0$. What about the 3cell? look at the degree in 2-cell. If we look at the attachment across the equator, we see that the two faces are attached oppositely so the degree is again 0. So the picture is symmetric in all three faces so $\partial_3 = 0$. So

$$C_*^{CW}(T^3) = 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

So

$$H_n(T^3) = \begin{cases} \mathbb{Z} & n = 3, 0 \\ \mathbb{Z}^3 & n = 1, 2 \\ 0 & \text{else} \end{cases}$$

Much easier than if we were trying to chop up into simplicial/ singular homologies.
Some general properties if X is CW complex

1. If X is n dimensional then $H_i(X)$ for $i > n$
2. If X has no $i-$cells then $H_i(X) = 0$
3. If X has k $i-$cells then $H_i(X)$ is generated by at most k elements.
4. If X has no two of its cells in adjacent dimensions, then $H_i(X)$ is free abelian and generated by its cells, for all i. In this case, it is $C_n(X) \to 0 \to C_{n-2}(X) \to 0 \to C_{n-2}(X) \to 0 \ldots$. Because ker/im is everything / 0 so it works out.

Example $\mathbb{C}P^n$ was built out of 1 even dimensional cell in each even dimension up to $2n$.
Then $C_*^{CW}(\mathbb{C}P^n) = \mathbb{Z}^{2n} \to 0^{2n-1} \to \mathbb{Z}^{2n-2} \ldots \mathbb{Z}^2 \to 0^1 \to \mathbb{Z}^0$. So

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if i is even and } i \leq 2n \\ 0 & \text{else} \end{cases}$$

Note there is a space $\mathbb{C}P^\infty$ with homology $\mathbb{Z}$ in every even dimension.

0.6 Making spaces with given homology

Recall that the map $f_n : S^1 \to S^1$ given by $f(z) = z^n$ is a degree n map. How do we get a degree n map?
Recall $\Sigma S^n = S^{n+1}$, (suspension, i.e. cross it with I and quotient the top, and bottom), and a map $f : X \to Y$ induces a map $\Sigma f : \Sigma X \to \Sigma Y$. To do this, we have map f from $X \times I$ to $Y \times I$, and now, we crush the top, and the bottom, similarly.
In MVS, for ΣS^{n-1}, we got

$$0 \to H_n(S^n) \xrightarrow{\partial_*} H_{n-1}(S^{n-1}) \to 0$$

Note that $\Sigma f_n : S^2 \to S^2$ respects the north/south hemi decomposition, so by the naturality of ∂_*, we get a commutative diagram as follows. I.e.

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\partial_*"] \arrow[d, "\Sigma f_n"] & H_{n-1}(S^{n-1}) \arrow[d, "f_n"] \\ H_n(S^n) \arrow[r, "\partial_*"] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

By commutativity, we have that Σf_* is degree n map. Iterating this construction shows that $\Sigma^{k-1} f_n : S^k \to S^k$ is a degree n map.
We can construct a space with $H_n(X) = \mathbb{Z}_m$ and $H_i(X) = 0$ for $i \neq 0, n$ by attaching an $n+1$ cell to S^n by the $n$cell to the 0cell. By a map of degree m, then

$$C_*^{CW} = 0 \to \mathbb{Z}^{\dim\, n+1} \xrightarrow{m}\to \mathbb{Z}^{\dim\, n} \xrightarrow{0} 0 \to \ldots \to 0 \to \mathbb{Z}^{\dim\, 0} \to 0$$

so we get the desired homology groups.

Definition 0.43: Let G be an abelian group and $n \geq 1$ be an integers. A Moore space $M(G, n)$ is a CW complex with $H_n(M(G, n)) = G$, $H_0(M(G, n)) = \mathbb{Z}$, $H_i(M(G, n)) = 0$ if $i \neq 0, n$. If $n > 1$, we also require $\pi_1(M(G, n)) = 0$.

For a finitely generated abelian group G by the funamental theorem of finitely generated abelian groups,

$$G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_l}$$

To construct $M(G, n)$, take $\bigvee_{i=1}^{l+k} S^n$ for X^n and attach l $n+1$ cells by maps of degree $d_1, \ldots, d_l$, to separate $N-$spheres.
Recall that $H_p(\bigvee_i X_i) = \oplus_i H_p(X_i)$, then given a list of groups $G_1, \ldots, G_l$, we can construct a space X with $H_i(X) = G_i$ by $\bigvee_{i=1}^{\ell} M(G_i, i)$.

Lecture 25. The Euler Characteristic

This is the first topological invariant discovered.
We consider CW compositions of S^2, we notice that $V - E + F = 2$ for all decompositions. Now, consider the CW decomposition of T^2, where $V - E + F = 0$ for all those decompositions.

Definition 0.44: Let X be a finite CW complex with a CW structure, with C_i $i-$cells. We define the Euler Characteristic, $\chi(X)$, to be

$$\chi(X) = \sum_{i=0}^{n} (-1)^i C_i$$

where $n = dim(X)$.

Natural question: Does this depend on the CW-structure?

Definition 0.45: Let $G = \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \ldots \oplus \mathbb{Z}_{d_l}$ be a finitely generated abelian group. Define $rank(G) = rk(G)$ to be the integer k above, which refers to the free abelian part.

Lemma 0.55:
How does rank behave in short exact sequence?
If

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups. Then $rk(B) = rk(A) + rk(C)$, where C is roughly like B/A.

Proof: Proof idea: Tensor the sequence with $\mathbb{Q}$, make torsion go away, and use rank-nullity theorem. Not sure about torsion and tensor with $\mathbb{Q}$. □

Theorem 0.56:

$$\chi(X) = \sum_{i=0}^{n=dim(X)} (-1)^i rk(H_i(X))$$

The point is that the sum on the RHS does not depend on X, so this is a topological space invariant.

Proof: The cellular homology chain complex looks like:

$$0 \to C_n^{CW}(X) \xrightarrow{d_n} C_{n-1}^{CW}(X) \xrightarrow{d_{n-1}} C_{n-2}^{CW}(X) \ldots C_1^{CW}(X) \to C_0^{CW}(X) \to 0$$

□

Let $Z_i = \ker d_i$ and $B_i = Im\ d_{i+1}$. Then we have a SES

$$0 \to B_i \to Z_i \to Z_i/B_i = H_i(X) \to 0$$

we also have another SES

$$0 \to Z_i \xrightarrow{i} C_i^{CW}(X) \xrightarrow{d_i} B_{i-1} \to 0$$

By our lemma, 1 implies $rk(Z_i) = rk(B_i) + rk(H_i)$, 2 implies $rk(c_i) = rk(z_i) + rk(B_{i-1})$. Sub 1 into 2, we get

$$rk(C_i) = rk(B_i) + rk(H_i) + rk(B_{i-1})$$

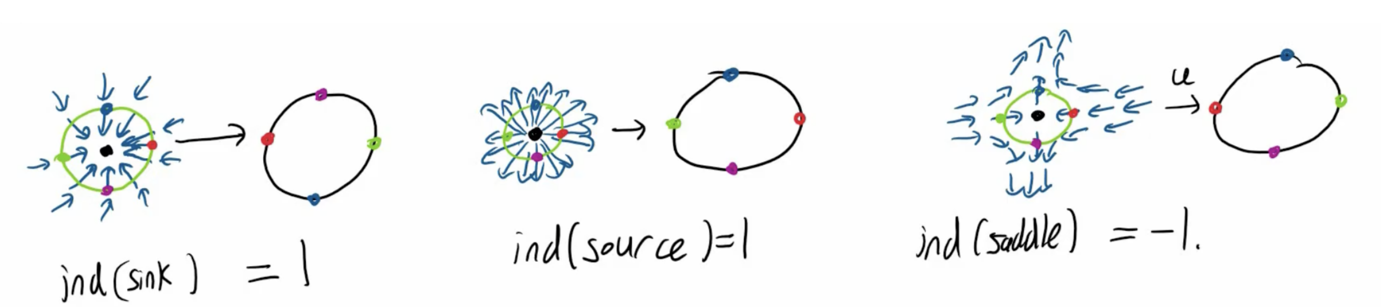
Taking an alternating sum get $\sum_i (-1)^i rk(c_i) = \sum_i (-1)^i rk(B_i) + rk(H_i) + rk(B_{i-1})$ In each adjacent summand, $rk(b_{i-1})$ appear with opposite signs. So they all cancel.
So $\sum_{i=0}^n rk(c_i) = \sum_{i=0}^n (-1)^i rk(H_i)$.
Quick examples

- $\chi(\Sigma_g)$ can be computed using polygon picture, Σ_g has 1 vertex, 2g edges, and 1 face. So $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$.
- $\chi(\#^k \mathbb{R}P^2) = 2 - k$.
- The only overlap is that $\chi(\Sigma_g) = \chi(\#^{2g}\mathbb{R}P^2)$, so surfaces are classificed by orientability and Euler characteristic.
- $$\chi(S^n) = \begin{cases} 0 & \text{n odd} \\ 2 & \text{n even} \end{cases}$$

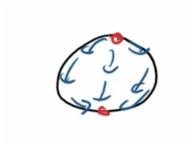
0.7 Euler Characteristics and vector fields

Recall that S^2 has no non-vanishing vector fields but it does have a vector field with 2 zeroes on it, that is, the vector field with 0 on the top and bottom, and have a flow going from the top to bottom.
Also $\chi(S^2) = 2$. So we observe that the spheres with euler characteristic of 2 has two zeroes, whereas the spheres (odd degree) has non vanishing vectorfields, and have EC of 0.
Also consider T^2, it has non vanishing vector field. (imagine the square with all vectors flowing the same way). Also $\chi(T^2) = 0$. So is this a coincidence?

Definition 0.46: Let V be a vector field on a differentiable manifold. (assignment of vector at each point). If z is an isolated 0 of the vector field, the we can look at D^n neighbourhood of z with no other zeroes. At each point of ∂D^n, we have a n-dim vector, which after rescaling, we can consider as a point in S^{n-1}. That is, we can scale each vector to the length of 1 in its direction. The only vector we cannot do so is 0, but since there are no zero vectors, we can apply this for every vectors. So we get a map $\partial D^n = S^{n-1} \to S^{n-1}$. (i.e. looking at the boundary, and the vectors on that boundary). Call this map U. Define $ind_Z(v) = deg(U)$.



So $\chi(S^2) = \sum_i ind_{z_i}(V)$ for V the vector field.



It has a source on top and a sink on the bottom. THey add up to 2.

Theorem 0.57 (Poincare Hopf index theorem):
Let X a differentiable manifold and let V be a vector field on X with isolated zeroes z_i. Then $\chi(X) = \sum_i ind_{z_i}(v)$.

Corollary 14: If $|\chi(X)| = n$, then any vector field has at least n zeroes. This generalizes the hairy ball theorem.

fun exercise: try drawing minimum number of zeroes on different surfaces.

Lecture 26. Homology with Coefficients

Let us first recall the ordinary definition of homology, which is with $\mathbb{Z}$ coefficients. (The normal homology). Take $\sum_i n_i\sigma_i$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ a singular n chain to be $C_n(X) = C_n(X, \mathbb{Z})$. Take homology of $C_*(X)$ to get $H_n(X) = H_n(X, \mathbb{Z})$. This is the homology with integer coefficients.

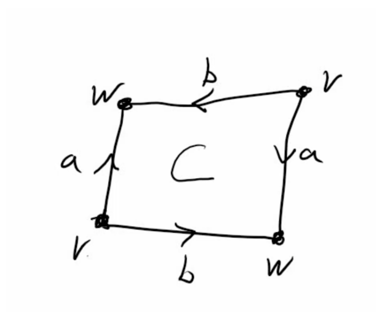
Definition 0.47: Let G be an abelian group, let $C_n(X, G)$ be elements of the form $\sum_i g_i\sigma_i$ and $\sigma_i : \Delta^n \to X$ is a singular n-simplex.
Like before, we can define $\partial g\sigma_i = \sum_j (-1)^j g(\text{faces of } \sigma_i)$.
So the coefficient of each face, Δ^n is either g or $-g$. Like before, $\partial^2 = 0$. So we obtain a chain complex, we call $C_*(X; G)$.

Definition 0.48: Define $H_*(X; G)$ is the homology of the complex $C_*(X; G)$. Note that we use abelian groups so that $Ker(\partial_n)/Im(\partial_{n+1})$ is a group. We assume abelian because this always ensure this subgroup is normal.
Most of the machinery developed works just fine with homology with coefficients.
For example, MVS works, cellular homology works, which is equal to the simplicial homology and the singular homology.

The questionn is, which coefficients do I use?

- $\mathbb{Z}_2$: simplicity, as signs disappear and orientation does not matter!
- $\mathbb{Q}$ torsion diappear. I am no sure what this means!
- $\mathbb{R}$ captures analytic behaviour and can be used for an algebraic approach to integration. The past two items are both under the umbrella of De Rham cohomology.

Today, we will focus on $\mathbb{Z}_2$ and see how it captures non-orientable information.
Example: Calculate $H_*(\mathbb{R}P^2; \mathbb{Z}_2)$
$\mathbb{R}P^2$ has the following cell decomposition:



So

- $C_0(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2^V \oplus \mathbb{Z}_2^W$
- $C_1(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2^a \oplus \mathbb{Z}_2^b$
- $C_2(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2^C$

$\partial_a = w - v, \partial_b = w - v, \partial_c = 2a - 2b = 0$.
How, we see that

- $H_2(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$ (big win), why, since $H_2(\mathbb{R}P^2, \mathbb{Z}) = 0$, so $\mathbb{Z}$ homology cannot detect 2-diml behaviour.
- $H_1(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$
- $H_0(\mathbb{R}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$

Now, here are some more general examples: $H_n(S^n; G) = G$ since the cellular complex for $n > 1$, it looks like

$$0^{n+1} \to G^n \to 0^{n-1} \to \dots$$

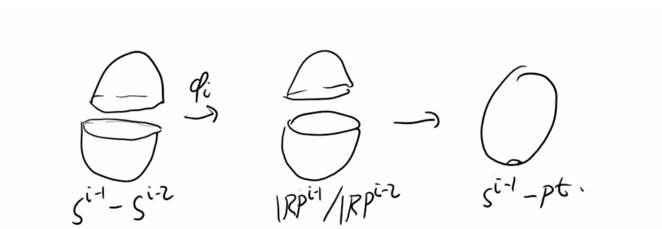
For computing maps in cellular homology complexes, the following is useful.

Lemma 0.58:
If $f : S^k \to S^k$ has degree m, then the induced map $f_* : H_k(S^k; G) \to H_k(S^k; G)$ is multiplication by M, as $(g \mapsto g + g + \ldots + g)$.

Recall that $\mathbb{R}P^n$ has a cell structure with an $i-$cell for each $0 \leq i \leq n$. The attachment map $\psi_i : \partial D^i = S^{i-1} \to \mathbb{R}P^{i-1}$ is the 2 sheeted covering map.
The boundary map on cellular homology is the degree map $\psi_i \circ q$ where we have

$$S^{i-1} \to \mathbb{R}P^{i-1} \xrightarrow{q} \mathbb{R}P^{i-1}/\mathbb{R}P^{i-2} = S^{i-1}$$

What happens when we trace out the top cell of S^{i-1}?



Get a homeo on the top and bottom hemisphere of $S^{i-1} - S^{i-2} \to S^{i-1} - \{pt\}$, the homeos differ by pre-comp with antipodal map. So $deg(\psi_i \circ q) = 1 + (-1)^i$. The $(-1)^i$ comes from flipping i coordinates.
So with $\mathbb{Z}$ coefficients, we have $C_*(\mathbb{R}P^n; \mathbb{Z})$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \ldots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0 \text{ , n is even}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \ldots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0 \text{ , n is odd}$$

So $H_k(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or if } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n \\ 0 & \text{else} \end{cases}$ So homology with integer coefficient sees things in the odd degrees, not even behaviour of $\mathbb{R}P^2$. On the other hand, $C_*(\mathbb{R}P^n; \mathbb{Z}_2)$ is equal to

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \ldots \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

It alternates between 0 and multiplication by 2. But everything multiply by 2 is the zero map.
So, see below, which helps us to see behaviours in the even degrees $H_k(\mathbb{R}P^n; \mathbb{Z}_2) \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n \\ 0 & \text{else} \end{cases}$

Definition 0.49: Let X be a topological space and $A \subset X$ a nonnempty closed subspcae so that there exists a NBD $N \subset X$ with N deformation retracting to A. Then (X, A) is said to be a good pair. I.e. there exists some rooms for A to expand into N and for N shrink back to A.

For example, if X is a CS complex and A is a subcomplex then (X, A) is a good pair.

Theorem 0.59:
If (X, A) is a good pair, then there exists a LES in homology of the form:

$$H_n(A; G) \xrightarrow{i_*} H_n(X; G) \xrightarrow{q_*} H_n(X/A; G) \xrightarrow{\partial_*} H_{n-1}(A; G)$$

where i_* is inclusion, q_* is quotient map, ∂_* is from zig zag lemma.

Apply this to $(\mathbb{R}P^2, \mathbb{R}P^1)$, we get

$$H_2(\mathbb{R}P^1; \mathbb{Z}_2) \xrightarrow{i} H_2(\mathbb{R}P^2; \mathbb{Z}) \xrightarrow{q} H_2(\mathbb{R}P^2/\mathbb{R}P^{1(=S^2)}; \mathbb{Z}_2)$$

$$0 \to \mathbb{Z}_2 \xrightarrow{q_*} \mathbb{Z}_2$$

So q_* is an injection and isomorphism. In particular, q is not null homotopic. It induces a nontrivial map on the second homology with $\mathbb{Z}_2$ coefficients.
Note that homology with $\mathbb{Z}$ coefficients cannot detect this map. The reason being none of the homology groups line up. At $H_1, H_1(S^2) = 0$ and $H_2(RP^2) = 0$, but $\mathbb{Z}_2$ coefficients do work.

Theorem 0.60 (Borsuk Ulam Theorem):
For every map $g : S^n \to \mathbb{R}^n$,there exists a pair of antipodal points X and $-X$ with $g(X) = g(-X)$. This is a surprising theorem.

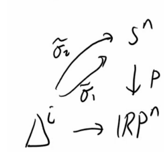
Most of the heavy lifting is in the following:

Proposition 0.61:
An odd map $f : S^n \to S^n$ (a map satisfying $f(-x) = -f(x)$) for all x has odd degree.

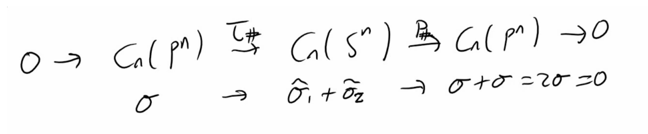
First, note that if f is odd, then f induces a map $\overline{f} : RP^n \to RP^n$. (Need $f(-x) = f(x)$, have $f(-x) =^{odd} -f(x) =^{RP^n} f(x)$)
we have the following commutative diagram. So $\overline{f} \circ p = p \circ f$.

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "p"'] & S^n \arrow[r, "p"] & \mathbb{R}P^n \\
\mathbb{R}P^n \arrow[ur] \arrow[urr, "\overline{f}"'] & &
\end{tikzcd}$$

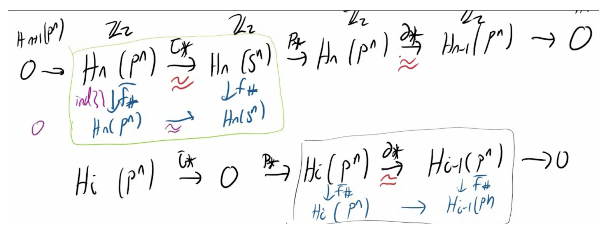
A singular i simplex is a map, by lifting criterioin, there exists a lift $\sigma : \Delta^i \to RP^n$. We get two lifts $\widetilde{\sigma_2}, \widetilde{\sigma_1}$ because there are two chocies depending on where it gets lifted to.



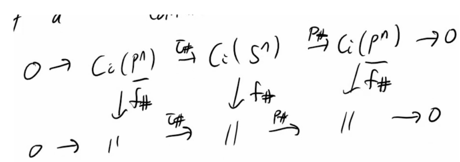
We write P^n for $\mathbb{R}P^n$ and everything is in $\mathbb{Z}_2$.
Let $\tau : C_n(P^n) \to C_n(S^n)$ be $\tau(\sigma) = \widetilde{\sigma_1} + \widetilde{\sigma_2}$.
We then get a SES



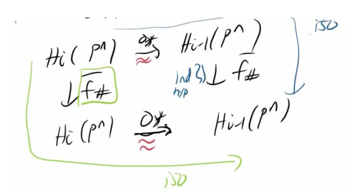
Proof: The SES of chain complexes leads to LES in H_* which looks like the following:



where the ∂ come from the zig zag lemma. Note that τ_* is an isomorphism. And ∂_* is surjection, and isomorphism. Similarly for the bottom level. Now consider the $f, \overline{f}$ maps.
We get a commutative diagram



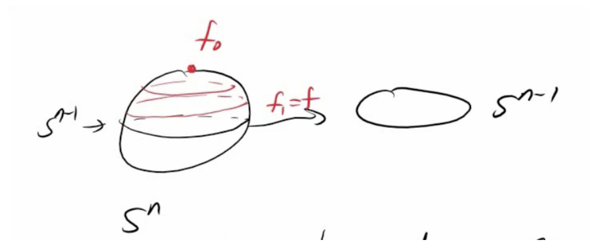
So by naturality of ∂_*, we get a chain map of the LES to itself. The point is all these maps commute.
Goal: $\overline{f_\#}$ is isomorphism in all dimensions. We do this by inductoin. Base case is $i = 0$ is easy, because the map preserves connected components.
Now, suppose that this true for H_{i-1}, by the commutative square above,



The ∂_* are isomorphism in all degrees, and $\overline{f_*}$ is isomorphism by hypothesis. So by commutativity the $\overline{f_*}$ is also an isomorphism.
So we conclude $\overline{f_*}$ is an iso on all i.
Also by the other square, since it is isomorphism by induction, and there are also isomorphisms, so the leftover map is also isomorphism. So $f_* : H_n(S^n) \to H_n(S^n)$ must be an isomorphism. So f_* needs to be multiplication by a odd number. (otherwise u always get 0). So f has odd degree. $\square$

Proof: Proof of Borsuk Ulam

Let $g : S^n \to \mathbb{R}^n$ be a map. Suppose that $g(X) \neq g(-X)$ for all X. Define a map $f : S^n \to S^{n-1}$ by $f(x) = \frac{g(x)-g(-x)}{|g(x)-g(-x)|}$, then f is odd and so is $f\mid_{S^{n-1}}$ where S^{n-1} is equator. Then f must have odd degree. But we can continuously deform the map, so $f\mid_{S^{n-1}}$ is nullhomotopic, so it has degree 0. We get contradiction.



$\square$