PMath 464

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Contents

Algebraic Geometry Insights

(Chat with Prof David Jao)

- Main ideas for algebraic geometry

  Consider differential geometry, things are (rather) nice and (relatively) easy to visualize, because we are usually working with vectors in spaces like $\mathbb{R}^n$ for some finite n. Now, we want to do similar geometry stuff on more abstract algebraic objects, such as fields, maximal ideals and prime ideals, and even more complicated ones. It is obviously much harder to visualize them!

  But what mathematicians do is to make alternative definitions for the algebraic objects in order to do geometry on them. For example, Zariski tangent space on algebraic sets as the usual sense tangent space on usual sets. So the main point of algebraic geometry is to make algebraic objects nicer in order to do geometry on them. Algebraic geometry makes nice definitions on algebraic objects, so that the geometry operations would carry out. This is done in the sense that if you're working with those definitions for object in $\mathbb{R}^n$, you get the equivalent result as usual (as in differential geometry), but if you're working with algebraic objects, things are still elegant and make sense.

  One line summary: In algebraic geometry, mathematicians define definitions and theorems in order to do geometry (i.e. differential geometry sense) on abstract algebraic objects.

- Zariski tangent space:

  Why is the zariski tangent space defined as $T_P(V) = (M/M^2)^*$?

  Why taking the dual?

  Direction derivative is the inner product. For example, we want some object that are of the form

$$V^* = \{f : W \to F \mid \exists u \in U, \forall u, f(v) = \langle u, v\rangle\}$$

  So for $\phi \in M/M^2$, we take an element $f \in (M/M^2)^*$, where $f(\phi) = 0$ implies orthogonality. Just as in the linear algebra/ functional analysis/ differential geometry sense.

  Why taking the M/M^2?

  We are taking the maximum ideals, and "discarding" the elements in M^2. In some sense, the ideal M^2 represents upper derivatives. What we want to do is to only look at the vectors at the "current tangent space" level. So we only look at the "constant terms", which is elements in M, removing the "upper dimensional derivatives". This is similar to how evaluating $x = a$ for this taylor series only ourputs the constant term, and ignores all the upper level derivatives.

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

- Projective closure vs algebraic closure

  Why projective closure?

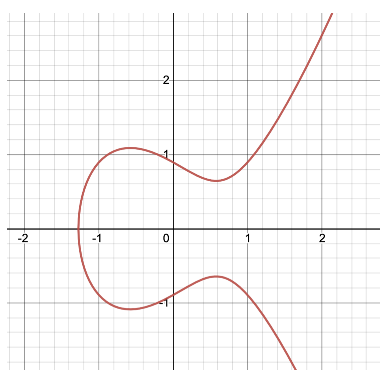
  Again, here's an example on how projective closure identifies "all points of intersection".

  Consider Bezout's theorem.

  "If two plane algebraic curve of degrees d_1 and d_2 have no component in common, then they have $d_1 d_2$ intersection points, counted with their multiplicity, and including points at infinity and points with complex coordinates."

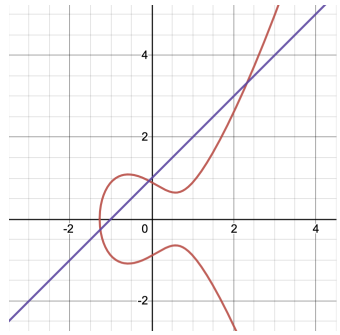
  Let's try a simple example: consider the intersection point of $E : y^2 = x^3 + ax + b$ and $L = mx + ny + c$. By Bezou't theorem, there should be $3 \cdot 1$ points in common.

  Consider the following, elliptic curve in $\mathbb{R}^2$.

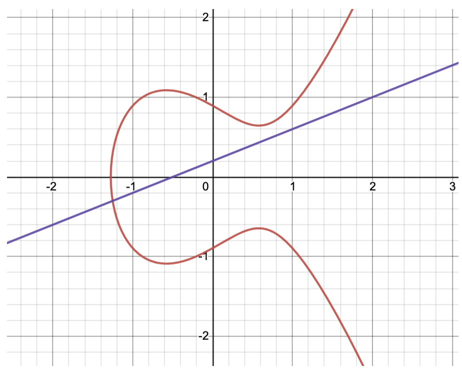


  <u>A normal situation</u>

  Here are three intersection points

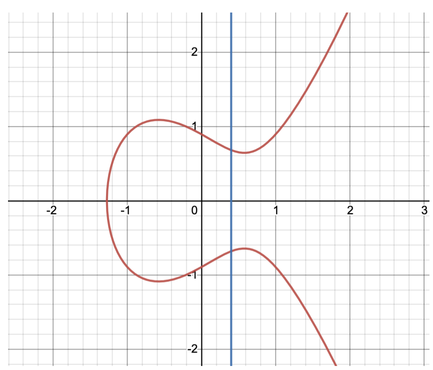


<u>A situation we need algebraic closure</u>



The two other roots are in $\mathbb{C}^2$ which is the algebraic closure of $\mathbb{R}^2$.

<u>A situation where we need projective closure</u>



In this case, even if you look in $\mathbb{C}^2$, you can't find the intersection point.

So, mathematicians used projective spaces to find the other point. The third intersection point is actually denoted $\mathcal{O}$, which is a point at infinity. This point is in the projective closure of these two curves.

- Scheme is defined as the set of prime ideals, where you can see the prime ideals as points. This is another definition that makes the theory more elegant.

1 Algebraic Geometry Week 1

1.1 Algebraic Sets

Definition 1.1 (2.1): Let n be a positive integer. Affine space $\mathbb{A}^n$ is the set $\mathbb{C}^n$.

Definition 1.2 (2.2): Let S be a subset of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The algebraic set corresponding to S is the set:

$$V(S) = \{x \in \mathbb{A}^n \mid f(x) = 0, \forall f \in S\}$$

In other words, $V(S)$ is the set of points where all the functions in S vanish.

Definition 1.3 (2.3): Let $X \subset \mathbb{A}^n$ be a subset of affine space. The ideal of X is the set

$$I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n \mid f(P) = 0, \forall P \in X\}$$

In other words, $I(X)$ is the set of polynomials that vanish on all of X.

Theorem 1.1 (2.4):
Let X be a subset of $\mathbb{A}^n$, $I(X)$ the ideal of X. Then $I(X)$ is an ideal of the ring $\mathbb{C}[x_1, \ldots, x_n]$. Moreover, $I(X)$ is a radical ideal: if $f^n \in I(X)$ for some positive integer n, then $f \in I(X)$.

Proof: To check the set is an ideal, we need to check it is an additive subgroup and it is closed under multiplication by elements in the ring.
I.e. show $f, g \in I(X) \implies f \pm g \in I(X)$, $0 \in I(X)$, and any element h in $\mathbb{C}[x_1, \ldots x_n$ we have $hf \in I(X)$.
To show radical ideal, if $f^n \in I(X)$, then $(f(x))^n = 0, \forall x \in I(X)$, so $f(x) = 0$ for all $x \in I(X)$. Hence $f \in I(X)$. □

Note that every algebraic set is defined by a finite set of polynomials. By Hilbert Basis Theorem, that says every ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated. So if $I(X) = (f_1, \ldots, f_r)$ then $X = V(f_1, \ldots, f_r)$.

Definition 1.4 (2.5): An ideal I of a ring R is called radical if every $r \in R$ with $r^n \in I$ for some positive integer n satisfies $r \in I$. In other words, I is closed under radicals. For an arbitrary ideal I, of a ring R, define the radical of I to be

$$Rad(I) = \{r \in R \mid r^n \in I \text{ for some integer } n > 0\}$$

Theorem 1.2 (2.6):
Let I be an ideal of a ring R. Then $rad(I)$ is a radical ideal of R containing I.

Proof: Containing I is obvious, take $n = 1$.
We will just show $rad(I)$ is radical ideal.
It contains 0 is clear. Now, if $r \in rad(I), r \in R$, then $j^n \in I$, $(rj)^n = r^n j^n \in I$, so $rj \in rad(I)$.
If $j^n \in rad(I)$ then $(j^n)^m \in I$, $\implies j^{nm} \in I$ so $j \in rad(I)$. This shows $rad(I)$ is radical.
Last to do is to show closure under addition, subtraction. Say $j_1, j_2 \in rad(I)$. Then $j_1^{n_1}, j_2^{n_2} \in I$. Then consider $(j_1 \pm j_2)^{n_1+n_2}$. When it is expanded, every term in the expanded product will either contain factor of $j_1^{n_1}$ or $j_2^{n_2}$ so all terms lie in I. So $(j_1 \pm j_2)^{n_1+n_2} \in rad(I)$. □

Theorem 1.3 (2.7 Hilbert's Nullstellensatz):
Let n be a positive integer. There is a bijection

$$\{\text{algebraic subsets of } \mathbb{A}^n\} \iff \{\text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

The bijection is given by $X \mapsto I(X), I \mapsto V(I)$.

Note forany $X \subset \mathbb{A}^n$, $I(X)$ is radical ideal, so that is why we restrict to radical ideals. Also $V(I)$ is alwasy an algebraic set by definition. We also have $V(I(X)) = X$ for every algebraic set X.
Proving $I(V(I)) = I$ is hard though, we will see online proofs.
Here are some useful notes:

- Bigger ideals correspond to smaller algebraic sets. $X \subset Y$ if and only if $I(Y) \subset I(X)$.
- Union of algebraic sets correspond to intersection of ideals:

  $$I(X \cup Y) = I(X) \cap I(Y)$$

  TO see why, say f vanishes on X and Y, that is, $f \in I(X \cup Y)$, then it is simultaneously in $I(X), I(Y)$. If $f \in I(X) \cap I(Y)$, f vanishes on both X, Y, also their union.
- If $I(X)$ is a maximal ideal, the algebraic set it correspond to is a point. (Proof see page 9 in W1 notes).
- Best way to show an ideal is maximal is to mod out and showing quotient is a field.
- Best way to show R/I is a field is to find an onto homomorphism of R to a field, such that its kernel is I.
- $I(X)$ is a maximal ideal if and only if X is a single point.
- If $I(X)$ is not prime ideal, then X can be written as union of two proper algebraic subsets.
- $I(X)$ is not prime if and only if X is the union of two proper algebraic subsets.

Definition 1.5 (2.8): Let X be a nonempty algebraic set. We say that X is reducible if and only if it is the union $X = Y_1 \cup Y_2$ of two proper algebraic subsets.
We say X is irreducible iff it is not reducible. If X is empty, it is not irreducible nor reducible.

Theorem 1.4:
$I(X)$ is a prime ideal if and only if X is an irreducible algebraic set.

2 Algebraic Geometry Week 2

Definition 2.1 (1.1.): Let $X \subset \mathbb{A}^n$ $Y \subset \mathbb{A}^m$ be algebraic sets. A polynomial map from X to Y is a funciton $\phi : X \to Y$ such that the coordinates of $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$ are all polynomials in the coordinates of $\mathbb{A}^n$.

Definition 2.2 (1.2.): Let X and Y be algebraic sets. A polynomial map $f : X \to Y$ is an isomorphism if and only if there is a polynomial map $g : Y \to X$ such that $f \circ g = id, g \circ f = id$.

The above means that an isomorphism is one to one and onto. But it is **NOT** true that a polynomial map (bijective) is an isomorphism. There is an example in assns.
Ideally, we want an algebraic helper for an algebraic set X that is invariant under isomorphisms. That is, isomorphic algebraic sets have isomorphic partners.

Definition 2.3 (1.3): Let $X \subset \mathbb{A}^n$ be an algebraic set, with ideal $I(X)$. The coordinate ring of X is the ring

$$\Gamma(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X)$$

which is the ring of polynomial maps from X to $\mathbb{A}$.

My understanding of this: The reason why we mod it out by the ideal is that any polynomial in the ideal would vanish the points in the algebraic set. So any two polynomials that vanish the sets are regarded as the same. So we can mod it out by this ideal.

Themain idea is: a polynomial map from $\Gamma(X) \to \mathbb{A}^1$ is a single polynomial in n variables. But do they agree? Note that f, g agree on X if and only if $f - g$ is identically 0 on X. But we already built the place for those polynomials are zero on X, which is exactly our ideal! So f, g agree on X if and only if they are congruent modulo $I(X)$.

Theorem 2.1 (1.4):
Let $X \subset \mathbb{A}^n$ be an algebraic set, with coordinate ring $\Gamma(X)$. Then there is an one-to-one correspondence between algebraic subsets of X and radical ideals of $\Gamma(X)$, given by

$$Y \mapsto I(Y) \mod I(X)$$

and

$$I \mapsto V(\overline{I})$$

where $\overline{I}$ denotes the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the elements of $\mathbb{C}[x_1, \ldots, x_n]$ that lie in I modulo $I(X)$.

Moreover, under this correspondence, points correspond to maximal ideals, and irreducible subsets of X correspond to prime ideals.

Definition 2.4 (1.5 Pullback): Let $\phi : X \to Y$ be a polynomial map of algebraic sets. The pullback of ϕ is the homomorphism

$$\phi^* : \Gamma(Y) \to \Gamma(X)$$

given by $\phi^*(f) = f \circ \phi$.

Theorem 2.2 (1.6):
Let X and Y be algebraic sets with coordinate rings $\Gamma(X)$ and $\Gamma(Y)$ respectively. For any homomorphism $\psi : \Gamma(Y) \to \Gamma(X)$, there is a polynomial map $\phi : X \to Y$ such that $\psi = \phi^*$.

Proof:
Say $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. IF we want to build a polynomial map $\phi : X \to Y$, then we need to find m polynomials $\phi_1, \ldots, \phi_m$ in n variables.
$\phi : X \to Y$, we can write ϕ as $(\phi_1, \ldots, \phi_m)$.
Note that ϕ_i is the *ith* coordinate of ϕ. Now, ϕ_i is just the ith coordiante of ϕ. Let x_i bte the polynomial that picks out the ith coordinate in $\mathbb{A}^m$. So we can write $\phi_i = \psi(x_i)$. That is, pick out the x_i coordiante first then apply ψ (i.e. pick out the ith coordinate in Y and then apply the pullback.). Make that the same as ϕ_i. It doesnt make sense in that $\psi(x_i)$ is not a polynomial, but instead an equivalence class of polynomials in the ring of polys.
We therefore pick a polynomial on $\psi(x_i)$....?
this is the key.... formula?
For each i, we pick a polynomial $\phi_i \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\psi(x_i) = \phi_i \pmod{I(X)}$. Now we define $\phi : X \to Y$ by

$$\phi(P) = (\phi_1(P), \ldots, \phi_m(P))$$

This ϕ is a polynomial map. If we have $f \in I(Y)$, then $\psi(f) = 0$, and we have

$$\phi^*(f(x_1, \ldots, x_m)) = (f(x_1\phi, \ldots, x_m\phi)) = f(\phi_1, \ldots, \phi_m)$$

Modulo $I(X)$, we have

$$0 = \psi(f(x_1, \ldots, x_m)) = f(\psi(x_1), \ldots, \psi(x_m)) = f(\phi_1, \ldots, \phi_m)$$

So we conclude that $\phi^*(I(Y)) \subset I(X)$. This means that ϕ^* is well defined from $\Gamma(Y)$ to $\Gamma(X)$. (That is, if it vanishes in $\Gamma(Y)$ then it would vanish in $\Gamma(X)$) So ϕ is well defined from X to Y. If $P \in X$, then $f(P) = 0$ for all $f \in I(X)$. So, for all $g \in I(Y)$, $g(\phi(P)) = [\phi^*(g)](P) = 0$.
FInally we check $\phi^* = \psi$. FOr any polynomial $p(x_1, \ldots, x_m)$ we have modulo $I(X)$:

$$\phi^*(p) = p \circ \phi = p(\phi_1, \ldots, \phi_m)$$

and

$$\psi(p) = \psi(p(x_1, \ldots, x_n)) = p(\psi(x_1), \ldots, \psi(x_n)) = p(\phi_1, \ldots, \phi_m)$$

You can switch the ψ, p around is because homomorphism....?

∎

Theorem 2.3 (1.7):
Let X and Y be algebraic sets, with coordinate rings $\Gamma(X)$ and $\Gamma(Y)$, respectively. Then X is isomorphic to Y if and only if $\Gamma(X)$ is isomorphic to $\Gamma(Y)$.
Moreover, if $\phi : X \to Y$ is an isomorphism, then for any algebraic subset $V \subset Y$ then

$$I(\phi^{-1}(V)) = \phi^* I(V)$$

Note that $\phi^* I(V)$ is an ideal of $\Gamma(X)$ because ϕ^* is an isomorphism.

3 Algebraic Geometry Week 3

Note: recall that an algebraic set X is irreducible if and only if its corresponding ideal is prime, which is true if and only if its corresponding ring $\Gamma(X)$ is a domain. (aka rings with no zero divisors.)
Note that an algebraic set X is always a union of finitely many irreducible algebraic sets. This union is unique. (To show this, for any reducible set X, write it as union of two proper subvarieties. If they are reducible, write them again. Note that this process always stops. If it does not stop, we would have an infinite descending chain of varieties: $X \supsetneq X_1 \supsetneq \ldots$. This translates to an ascending chain of ideas. $I(X) \subset I(X_1) \subset \ldots$. This is impossible because $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian.)

Definition 3.1 (1.1.): Let X be an algebraic set. Write $X = X_1 \cup \ldots \cup X_r$ as union of finitely many irreducible algebraic subset. THen X_i are irreducible components of X.

Definition 3.2 (1.2.): A variety is an irreducible algebraic set.

Definition 3.3 (1.3.): Let X be a variety. The function field $K(X)$ of X is the fraction field of the coordinate ring $\Gamma(X)$. An element of $K(X)$ is called a rational function.

Definition 3.4 (1.4.): Let f be rational function on a variety X. Let $P \in X$ be a point. then f is defined at P if and only if there exists expression

$$f = \frac{p}{q}$$

for $p, q \in \Gamma(X), q(P) \neq 0$. If f is not defined at P, then P is a pole of f or f has a pole at P.
Note that there is no best representation for any function. You would have to use different versions for different points.

Theorem 3.1 (1.5):
Let f be a rational function on a variety X. $P \in X$ be a point. If there is a representation

$$f = \frac{p}{q}$$

for which $q(P) = 0, p(P) \neq 0$, then P is a pole of f.

Proof: Choose any representation $f = a/b, a, b \in \Gamma(X)$. Then $a/b = q/p$ yields $qp = bq$. Since $q(P) = 0, (bp)(P) = 0$. Since $p(P) \neq 0$, $b(P) = 0$. So denominator of f is always zero at P. So P is a pole.
The only way you have a "stealth" nonpole is both denominator and numerator are zero. □

Definition 3.5 (1.6): Let X be a variety, $P \in X$. The local ring at P is the ring

$$O_P(X) = \{f \in K(X) \mid f \text{ is defined at } P\}$$

It's pretty easy to show $O_P(X)$ is subring of $K(X)$. A local ring is a ring with unique maximal ideal. That is the set of nonunits is an ideal. Then we would realize that $O_P(X)$ is also a local ring whose maximal ideal $M_P(X)$ is:

$$M_P(X) = \{f \in K(X) \mid f = \frac{a}{b}, a(P) = 0, b(P) \neq 0\}$$

Note that a unit of $O_P(X)$ is a rational function whose reciprocal is also in $O_P(X)$. The units are the ones where the denominator does not vanish at P, nor the numerator. If the numerator vanish, by shortcut theorem, the reciproval would have a pole at P.

We also have

$$M_P(X) = I(P)O_P(X)$$

4 Algebraic Geometry Week 4

Definition 4.1 (0.1): Let V be an algebraic variety. A zariski closed subset of V is an algebraic subset of V. An subset $U \subset V$ is zariski open if $V - U$ is zariski closed.

Definition 4.2 (0.2): Let V and W be varities. A rational map from V to W is a function $f : U \to W$ for some nonempty Zariski open subset $U \subset V$, such that for every point $P \in U$, there are rational functions $f_1, \ldots, f_r$ on V, all defined at P such that

$$f(Q) = (f_1(Q), \ldots, f_r(Q))$$

for all Q such that $f_1(Q), \ldots, f_r(Q)$ are all defined.
A rational map is said to be defined at $P \in V$ if athere are rational functions $f_1, \ldots, f_r$ on V, all defined at P such that

$$f(1) = (f_1(Q), \ldots, f_r(Q))$$

, for all Q such that $f(Q), f_1(Q), \ldots f_r(Q)$ are all defined.
A rational map is a morphism on a subset $V' \subset V$ if it is defined at every point of V'.

Definition 4.3 (0.3): Let $U \subset V$ be a Zariski open subset. Then the ring of functions on U is the ring

$$\Gamma(U) = \{f \in K(V) \mid f \text{ has no poles in } U\}$$

Definition 4.4 (0.4): Let $\phi : V_1 \to V_2$ be a morphism of varieties, and $U_i \subset V_i$ be zariski open subsets such that $\phi(U_1) \subset U_2$. Then there is a $\mathbb{C}$ algebra homomorphism $\phi^* : \Gamma(U_2) \to \Gamma(U_1)$ defined by $\phi^*(f) = f \circ \phi$.

Theorem 4.1 (0.5):
Let V be an affine variety, $U \subset V$ a nonempty Zariski open subset.
If $U = V - V(f)$ for some $f \in \Gamma(V)$. Then

$$\Gamma(U) = \Gamma(V)[1/f] = \{p/f^r \mid r \in \mathbb{Z}, p \in \Gamma(V)\}$$

Theorem 4.2 (0.6. Krull's Hauptidealsatz):
Let X be a variety of dimension n, and $f \in \Gamma(X)$ a non-constant function. Then every irreducible componennt of the algebraic set $V(f) \subset X$ has dimension $n - 1$.

Theorem 4.3 (0.7):
Let V and W be two varieties, $\phi : V \to W$ a rational map. $P \in V$ a point where ϕ is defined. Then ϕ^* introduces an morphism from $O_{\phi(P)}(W)$ to $O_P(V)$ given as usual by $\phi^*(p) = p \circ \phi$.

Definition 4.5 (0.8): A rational map $f : V \to W$ is birational if and only if there is a rational map $g : W \to V$ such that $f \circ g$ and $g \circ f$ are both defined, and both equal to the identity function whenever they are defined.
A map f is said to be dominant if and only if there is no proper closed subset $Y \subset W$ such that $f(V) \subset Y$.

Theorem 4.4 (0.9):
Let $f : V \to W$ be a dominant rational map of varieties. Then there is a $\mathbb{C}$ algebra homomorphism $f^* : K(W) \to K(V)$ of function fields defined by

$$f^*(p) = p \circ f$$

for all $p \in K(W)$. Moreover, f is birational if and only if f^* is an isomorphism.

Theorem 4.5 (0.10):
Let $f : V \to W$ be a birational map with $f(P) = Q, f^{-1}(Q) = P$. Then $f^* : K(W) \to K(V)$ induces an isomorphism from $O_Q(W)$ to $O_P(V)$.

Definition 4.6 (0.11): Let A be a domain. P a prime ideal of A. K the fraction field of A. The localization of A at P is the set

$$A_P = \{\frac{a}{b} \mid a, b \in A, b \notin P\}$$

Theorem 4.6 (0.12):
The localization of A at P is a local ring. Recall that a local ring is a ring with a unique maximal ideal; that is, the set of all non-units is an ideal.

Theorem 4.7 (0.13):
Let A be a noetherian domain, P a prime ideal of A. Then localization A_P is noetherian.

5 Algebraic Geometry Week 5

Definition 5.1 (1.1): Let V a variety, let

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = V$$

be a chain of maximal length s.t. each V_i is a variety. Then the dimension of V is equal to n. Empty set does not have a dimension.

Definition 5.2 (1.2): Let D be a domain. Let

$$V_0 \supsetneq P_1 \supsetneq \ldots \supsetneq P_n = (0)$$

be chain of maximal length such that each P_i is a prime ideal of D. Then the Krull dimension of D is n. If no such maximal chain exists, then the Krull dimension of D is infinite.

Theorem 5.1 (1.3):
The dimension of $\mathbb{A}^n$ is n.

Strategy to determine the dimension of an algebraic set.

1. Check if $V \subset \mathbb{A}^n$ is empty. If it is, we're done.
2. Find a maximal chain of varieties $V_0 \subsetneq \ldots \subsetneq \mathbb{A}^n$ that has V in it.
3. This involves showing that each inclusion $V_i \subsetneq V_{i+1}$ is strict (find a point of V_{i+1} that isn't in V_i).
4. This also involves showing that each V_i is irreducible.
5. Once you've done all that, find the m for which $V = V_m$, and conclude that $\dim V = m$.

Definition 5.3 (1.3): Jacobian matrix: a matrix whose rows are $\nabla f_i(P)$.

Definition 5.4 (2.2): Let P be a point on a variety $V = V(f_1, \ldots, d_m) \subset \mathbb{A}^n$. Then P is a smooth point of V if and only if tha rank of $J_V(P)$ equal $n - dimV$. If P is not a smooth point of V then it is a singular point of V.

Theorem 5.2 (2.3):
Let $V = V(0)$ be a subvariety of $\mathbb{A}^n$. Let $P \in V$ be a point. Then $dimV = n - 1$. And V is smooth at P if and only if $\nabla(f)(P) \neq 0$.

Theorem 5.3 (2.4):
Let P be a point on a variety $V \subset \mathbb{A}^n$. Let $M = M(P) \subset \Gamma(V)$ be the maximal ideal corresponding to P. Then

$$dim(M/M^2) + rank(J_V(P)) = n$$

where the dimension on the left hand side is the dimension as a vector space over $\mathbb{C}$. In particular, the rank of the Jacobian matrix does not depend on the choice of genrators for the ideal of V.

6 Algebraic Geometry Week 6

The Zariski tangent space.
The rowspace of Jacobian matrix is the span of $\nabla f_i(P)$. This is perpendicular to the nullspace of the Jacobian matrix, which we just hsowed was isomorphic to M/M^2.

Definition 6.1 (1.1): Let $V \subset \mathbb{A}^n$ be a variety, $P \in V$ a point. Let $\mathcal{O}_P(V)$ be the local ring at P, and let $\mathcal{M} = \mathcal{M}_P(V)$ be the maximal ideal of $\mathcal{O}_P(V)$. The zariski tangent space to V at P is

$$T_P(V) = (\mathcal{M}/\mathcal{M}^2)^*$$

This is, $T_P(V)$ is the dual $\mathbb{C}-$vector space to $\mathcal{M}_P(V)/\mathcal{M}_P(V)^2$.
The tangent space to V at P is the set

$$T_P(V) = P + kerJ_P(V) = \{Q \in \mathbb{Q}^n \mid Q - P \in kerJ_P(V)\}$$

where $J_P(V)$ is the Jacobian matrix of V at P.

Theorem 6.1 (1.2):
Let D be a domain, D_M the localization of D at a maximal ideal M. For any positive integer n, the natural inclusion $M \hookrightarrow \mathcal{M}$ includes an isomorphism

$$M^{n-1}/M^n \cong \mathcal{M}^{n-1}/\mathcal{M}^n$$

where $\mathcal{M}$ denotes the ideal of D_M generated by M. Particularly, $n = 1, n = 2$ we have

$$D/M \cong D_M/\mathcal{M}$$

and

$$M/M^2 \cong \mathcal{M}/\mathcal{M}^2$$

7 Algebraic Geometry Week 7

The projective space...

Definition 7.1 (1.1): Let n be a positive integer. Complex projective space $\mathbb{P}^n$ is the set of nonzero $(n+1)-$tuples of complex numbers, modulo the equivalence that $V \sim W$ if and only if $v = \lambda w$ for some $\lambda \in \mathbb{C}$.

For example, $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
Similar to $\mathbb{P}^1$, there is a copy of $\mathbb{A}^2$ sitting inside $\mathbb{P}^2$, namely $(x, y) \mapsto [x : y : 1]$. Any points $[x : y : z]$ with $z \neq 0$ corresponds to the point $(x/z, y/z)$ in $\mathbb{A}^2$. So we can disect this field as $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$.
Note that projective $n-$ space, $\mathbb{P}^n$ is just $n+1$ copies of $\mathbb{A}^n$ glued together, and in each case, $\mathbb{P}^n - \mathbb{A}^n$ is just a copy of $\mathbb{P}^{n-1}$.

Definition 7.2 (1.2): An algebraic subset of $\mathbb{P}^n$ is a subset $X \subset \mathbb{P}^n$. Such that for all i, $X \cap U_i$ is an algebraic subset of $U_i \cong \mathbb{A}^n$.

Definition 7.3 (1.3): A polynomial $f(X_0, \ldots, X_n)$ is homogeneous if and only if every term of f has the same degree.

Theorem 7.1 (1.4):
A subset $V \subset \mathbb{P}^n$ is algebraic if and only if it is the zero set $V(F_1, \ldots, F_r)$ of a finite set of homogeneous polynomials F_i.

Definition 7.4 (1.5): The irrelevant ideal of $\mathbb{C}[X_0, \ldots, X_n]$ is the ideal $(X_0, \ldots, X_n)$.

Theorem 7.2 (1.6):
Let n be a positive integer. There is a bijection between set of algebraic subsets of $\mathbb{P}^n$ and RRH ideals of $\mathbb{C}[x_0, \ldots, X_n]$. RRH stands for "relevant radical homogeneous". The bijection is given by $X \mapsto I(X)$ and $I \mapsto V(I)$.
Note that the irrelevant ideal is the only radical ideal left out of this correspondences. This is because its zero set is empty, despite not being a unit ideal. There are no points of $\mathbb{P}^n$ with all coordiantes zero!

Definition 7.5 (1.7):
A nonempty projective algebraic set is reducible if and only if it is the union of two proper projective algebraic subsets. It is irreducible if it is not reducible. An empty set is neither.

Theorem 7.3 (1.8):
A projective algebraic set V is irreducible if and only if the ideal $I(V)$ is prime.

Definition 7.6 (1.9): Let $V \subset \mathbb{A}^n$ be an affine algebraic set, and consider $\mathbb{A}^n$ as the subset $x_0 \neq 0$ in $\mathbb{P}^n$. The proejective closure of V in $\mathbb{P}^n$ is defined to be the intersection W of all projective algebraic sets containing V.

Theorem 7.4 (1.10):
Let $V \subset \mathbb{A}^n$ be an affine algebraic set. $W \subset \mathbb{P}^n$ be its projective closure. If $V = V(F)$ for some polynomial F of degree d in $\mathbb{C}[X_1, \ldots, X_n]$, then $W = V(f)$, where $f = x_0^d F(x_1/x_0, \ldots, x_n/x_0)$. So f is the homogenization of F and F is the dehomogenization of f.

Theorem 7.5 (1.11):
Let V be an affine algebraic subset, W its projective closure. Then $I(W)$ is the ideal generated by the homogenization of elements of $I(V)$.

8 Algebraic Geometry Week 8

Definition 8.1 (1.1. Morphism): Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be projective algebraic sets. A morphism from V to W is a function $f : V \to W$ such that every point $P \in V$, there is a $m+1$ tuple $[f_0 : f_1 \ldots : f_m]$ homogeneous polynomials of the same degree such that $f_i(P) \neq 0$ for some i and

$$f(Q) = [f_0(Q) : \ldots : f_m(Q)]$$

for all $Q \in V$ with $f_i(Q) \neq 0$ for some i.
An isomorphism is a morphism with an inverse morphism.

The most important exmaples of projective isomorphisms are projective change of coordinate.

Definition 8.2 (1.2): Let $V \subset \mathbb{P}^n$ be a projective variety. The function field $K(V)$ of V is the field $K(U)$ where U is any affine piece of V.
That is, U is the affine variety obtained by intersecting V with any standard affine piece of $\mathbb{P}^n$.
Moreover, if $P \in V$ is any point, then the local ring of V at P is the local ring $O_P(U)$ where U is any affine piece of V.

Definition 8.3 (1.3): Let V be a projective variety, $P \in V$ any point. Then P is a singular point of V if and only if P is a singular point of U, U being any affine piece of V. P is a smooth point of V othewise. If all points are smooth we call V smooth.

Definition 8.4 (1.4): Let P be a point on a projective variety V. The zariski tangent space to V at point p is the zariski tangent space to any affine piece V that contains P. Namely the dual of $M_P(V)/M_P(V)^2$.
The tangent space to V at P is the projective closure of the tangent space to V at P in any affine piece of V containing P.

Definition 8.5 (1.5): Let V be a projective variety. THe dimension of V is the dimension of any nonempty affine piece of V.
We don't care about homogeneous coordinate rings rn $(\mathbb{C}[X_0, \ldots, X_n]/I(V))$

Definition 8.6 (2.1): A curve is an algebraic variety of dimension one. A projective curve is a projective algebraic variety of dimension one, and an affine curve is an affine algebraic variety of dimension one.

Definition 8.7 (2.2): 2.2. A discrete valuation ring (or DVR, for short) is a Noetherian local ring whose maximal ideal is principal. A generator for the maximal ideal is called a uniformizing parameter, or uniformizer, for short.

Theorem 8.1 (2.3):
Let $C \subset \mathbb{A}^n$ be a curve,and $P \in C$ a smooth point. Then the local ring $O_P(C)$ is a DVR, and any linear function f whose zero set is not tangent to C at P is a uniformizer for $O_P(C)$.