PMath 453

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1 Section 1. Preliminaries

1.1 Review

Definition 1.1 (Inner Product):
(Note, when checking vector subspace, we must check $0 \in V$)
Let $\mathbb{F} = \mathbb{R}$, ro $\mathbb{C}$. Let U be a vector space over $\mathbb{F}$. An Inner product on U over $\mathbb{F}$ is a function with $\langle , \rangle : U \times U \to U$ with three properties:, $\forall u, v, w \in U, t \in \mathbb{F}$:

1. (Sesquilinearity): $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle, \langle tu, v\rangle = t\langle u, v\rangle$,
   $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle, \langle u, tv\rangle = \bar{t}\langle u, v\rangle$
2. (Conjugate symmetry): $\langle u, v\rangle = \overline{\langle v, u\rangle}$
3. (Positive definiteness): $\langle u, u\rangle \geq 0$ with $\langle u, u\rangle = 0 \iff u = 0$

Definition 1.2 (Inner product space, homomorphism or inner-product reserving): An Inner product space over $\mathbb{F}$ is a vector space $\mathbb{F}$ equipped with an innter product.
Given two innter product spaces U, V over $\mathbb{F}$, a linear map $L : U \to V$ is a homomorphism (or preserves inner product) if we have $\langle L(x), L(y)\rangle = \langle x, y\rangle, \forall x, y \in U$.

Definition 1.3 (norm): Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be a vector space over $\mathbb{F}$. A norm on U is a map: $\|:\| U \to \mathbb{R}$ such that $\forall u, v \in U$, $\forall t \in \mathbb{F}$, we have

- (Scaling): $\|tu\| = |t|\|u\|$
- (Positive definiteness): $\|u\| \geq 0$, with $\|u\| = 0 \iff u = 0$
- (Triangle inequality): $\|u + v\| \leq \|u\| + \|v\|$

u is a unit vector if $\|u\| = 1$. A normed linear space over $\mathbb{F}$ is a vector space over $\mathbb{F}$ equipped with a norm.
Given two normed linear spaces U, V over $\mathbb{F}$, a linear map $L : U \to V$ is a homomorphism of normed linear spaces if $\|L(x)\| = \|x\|, \forall x \in U$.

Theorem 1.1 (The norm induced by inner product):
Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be an inner product space over $\mathbb{F}$. For $u \in U$, we define $\|u\| = \sqrt{\langle u, u\rangle}$. Then, we have the following theorems:

1. (Scaling) $\|tu\| = |t|\|u\|$
2. (Positive Definiteness) $\|u\| > 0$ and $\|u\| = 0 \iff u = 0$
3. $\|u + v\|^2 = \|u\|^2 + 2Re\langle u, v\rangle + \|v\|^2$
4. (Pythagoras) If $\langle u, v\rangle = 0$ then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$
5. (Parallelogram) $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$
6. (Polarization Identity) if $\mathbb{F} = \mathbb{R}$ then $\langle u, v\rangle = \frac{1}{4}(\|u + v\| - \|u - v\|)$ and if $\mathbb{F} = \mathbb{C}$ then $\langle u, v\rangle = \frac{1}{4}\left(\|u + v\|^2 + i\|u + iv\|^2 = \|u - v\|^2 - i\|u - iv\|^2\right)$
7. (Cauchy-schwarz) $|\langle u, v\rangle| \leq \|u\|\|v\|$ with $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if $\{u, v\}$ is linearly dependent
8. (Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$

Note that $\|\|$ is a norm on U. That is, the inner product induces a norm.

Definition 1.4 (Metric): A metric on an nonempty set X is a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$:

1. (Positive Definiteness) $d(x, y) \geq 0$, and that $d(x, y) = 0 \iff x = y$
2. (Symmetry) $d(x, y) = d(y, x)$
3. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Definition 1.5 (Topology): A topology on set X is a set τ of subsets of X such that

- $\varnothing, X \in \mathcal{T}$
- if $U \in \mathcal{T}$ and $V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$
- if K is a set (index set) and $U_k \in \mathcal{T}$ for each $k \in K$ then $\bigcup_{k \in K} U_k \in \mathcal{T}$

Basically, a topology is a subset of the power set that holds some certain properties. The property being that the trivial subsets are both in the topology, and that **finite intersection** and **uncountable unions** are in the index set. For example, the sets open with respect to X form a topology. Note that the Borel σ algebra is defined in a similar manner, just with different rules.

For a subset $A \subseteq X$, we say A is open in X when $A \in \mathcal{T}$ and we say A is closed in X when $X \setminus A \in \mathcal{T}$. A set with a topology is called a topological space.
So, can a set have different topologies?
Note that

1. Given an inner product on a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, 1.1 shows that we get an associated norm on V, where $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in V$.
2. Given a norm on a vector space X, we can get an associated metric on X, where $d(x, y) = \|x - y\|$ for all $x, y \in X$.
3. Given a metric on a set X, we can define an associated topology on X by stipilating a susbset $A \subseteq X$

is open when it has the property that

$$\forall a \in A, \exists r > 0, r \in \mathbb{R}, \text{ such that } B(a,r) \subseteq A, \text{ where } B(a,r) = \{x \in X \mid d(a,x) < r\}$$

So, we conclude that inner product induces norm, which induces metric, which induces a topology.

Definition 1.6 (Sequences): Let $(x_N)_{n\geq 1}$ be a sequence in a metric space X. For $a \in X$, we say that the sequence (x_n) converges to $a \in X$, and $\lim_{n\to\infty} x_n = a$ if

$$\forall \epsilon > 0, \exists n \in \mathbb{Z}^+, (k \geq n) \implies d(x_n, a) < \epsilon$$

We say that x_n converges in X if it converges to some element $a \in X$. We say it's cauchy if

$$\forall \epsilon > 0, \exists n \in \mathbb{Z}^+, (k, l > n) \implies d(x_k, x_l) < \epsilon$$

Verify that if a sequence converges then it is cauchy.

Definition 1.7 (Complete, Banach space, Hilbert Space):
A metric space X is complete in X when every cauchy sequence converges. Note that if X is complete and $A \subseteq X$ is closed, then A is also complete. A complete normed linear space is a Banach Space. A complete inner product space is a Hilbert Space.
(Question, does hilbert give you banach?)

Definition 1.8 (Dense and separable): Let X be a topological space, $A \subseteq X$. A is dense if $\overline{A} = X$. In the case that X is a metric space, A is closed if, for every sequence $(x_n) \in A$ and every $a \in X$, if $x_n \to a$, then $a \in A$. A metric space is separable if it has a countably dense subset.

Example 1.2 (Vectors standard norms):
$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Standard inner product of $\mathbb{F}^n$ is $\langle x, y\rangle = y^*x = \sum_{k=1}^n x_k\overline{y_k}$. This inner product induces the standard norm, which is $\|x\|_2 = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}$. The standard norm induces the standard metric, which is $d(x,y) = \|x - y\|_2$. Since $\mathbb{F}^n$ is complete and separable under d_2 (Borel lebesgue?? Cant remember) so it is a finite dimensional separable hilbert space.

Example 1.3 (Space of sequences):
Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^\omega$ is the space of sequences $x = (x_1, \ldots) \in \mathbb{F}$. Let $\mathbb{F}^\infty$ be the space of eventual zero sequences. Note that it is countable dimensional with basis $\{e_1, e_2, \ldots\}$.

$$\ell_2 = \ell_2(\mathbb{F}) = \{x \in \mathbb{F}^\omega \mid \sum_{k=1}^n |x_k|^2 < \infty\}$$

We can define inner product on ℓ_2 by $\langle x, y\rangle = \sum_{k=1}^\infty x_k\overline{y_k}$, which induces 2-norm given by $\|x\|_2 = (\sum_{k=1}^\infty |x_k|)^{\frac{1}{2}}$, which induces the 2-metric, $d_2(x,y) = \|x, y\|_2$. ℓ_2 is complete and separable under d_2m so it's an infinite dimensional separable hilbert space.

Example 1.4 (L_2 norm):
Let $A \subseteq \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{M}(A) = \mathcal{M}(A, \mathbb{F})$ be the set of all measurable functions $f : A \to \mathbb{F}$. Let L_2 be the equivalences classes of measurable functions with finite ℓ_2 norm quotient by the equivalence relation of equivalence a.e..
So the standard inner product on $L_2(A)$ is $\langle f, g\rangle = \int_A f\overline{g}$. It induces the $2-$ norm given by $\|f\|_2 = (\int_A |f|^2)^{1/2}$, and the metric $d_2(f, g) = \|f - g\|_2$. Recall $L_2(A)$ is complete under d_2. (By Riesz-Fischer theorem). Also, if $a < b$, $L_2[a, b]$ is separable so it's an infinite dimensional separate hilbert space.

Example 1.5 (Vector $p-$norm):
Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x \in \mathbb{F}^n$, we define $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, 1 \leq p < \infty$. Define $\|x\|_\infty = \max\{|x_k|, | 1 \leq k \leq n\}$. Note that $1 \leq p \leq \infty$, $\|x\|_p$ gives a norm, and induces a metric d_p. THe infinity norm is the supremum norm, and the infinity metric is the supremum metric. Since $\mathbb{F}^n$ is complete and separable under d_p, $1 \leq p \leq \infty$ it's a finite dimensional separbale Banch space.

Example 1.6 (Sequence $p-$norm, and the ℓ_p spaces of sequences):
Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x \in \mathbb{F}^\omega$, we define $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, 1 \leq p < \infty$. Define $\|x\|_\infty = \sup\{|x_k|, k \in \mathbb{Z}^+\}$. ℓ_p are sequences with respective finite $p-$norms.
Note that $1 \leq p \leq \infty$, $\|x\|_p$ gives a norm, and induces a metric d_p. The infinity norm is the supremum norm, and the infinity metric is the supremum metric.
Note that ℓ_p is complete under $d_p, 1 \leq p \leq \infty$, so it is a Banach space. Note that ℓ_p separable when p is finite, and not separable when $p = \infty$.

Example 1.7 ($p-$ norm on Lp spaces):
Recall the L_p norms. (Or the supremum norm). These norms induce a $p-$ metric. Note that $L_p(A)$ is complete under $d_p, 1 \leq p \leq \infty$, so it is a Banach space. Recall $a < b$, $L_p[a, b]$ separable for p finite but not separable otherwise.

Theorem 1.8 (Holder):
p, q conjugate indices. $A \subseteq \mathbb{R}$ measurable.

- $\forall x, y \in \mathbb{F}^n, \forall x, y \in \mathbb{F}^\omega$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$.
- $\forall f, g \in \mathcal{M}(A), \|fg\|_1 \leq \|f\|_p \|g\|_q$.

Theorem 1.9 (Minkowski):
$p \in [1, \infty], A \subseteq \mathbb{R}, A \in \mathcal{M}$

- $\forall x, y \in \mathbb{F}^n, \forall x, y \in \mathbb{F}^\omega$, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$
- $\forall x, y \in \mathcal{M}(A), \|f + g\|_p \leq \|f\|_p + \|g\|_p$

Example 1.10:
Let X be a metric space. Fix $\mathbb{F}$ to be $\mathbb{C}, \mathbb{R}$. $\mathcal{F}(X)$ denote the spaces of all function $f : X \to \mathbb{F}$. $\mathcal{F}_b(X)$ be space of bounded functions. $C(x)$ space of continuous functions. $C_b(x)$ space of bounded continuous functions. Note that $\mathcal{F}_b(X)$ is a Banach space under the supremum metric, because convergence in $\mathcal{F}_b(X)$ under sup norm is same as uniform convergence on X, and $\mathcal{F}_b(X)$ is complete because uniform limit of continuous functions is continuous. If X is compact, then $C_b(X) = C(X)$, so $C(X)$ is Banach space. If $a, b \in \mathbb{R}, a < b, C[a, b]$ is separable by weierstrass polynomial approximation theorem.

1.2 Bounded Linear Operators

Definition 1.9 (Linear operator, transformation, functional):
U, V are normed linear spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, then a linear map F

1. $F : U \to V$ is a linear transformation or linear operator
2. $F : U \to U$ is a linear operator on U
3. $F : U \to \mathbb{F}$ is a linear functional on U

Definition 1.10 (Operator norm): Let U, V be normed linear spaces. $F : U \to V$ be a linear operator. The operator norm of F is:

$$\|F\| = \sup\{\|Fx\| \mid x \in U, \|x\| \le 1\}$$

F is bounded if it has finite operator norm. That is, when the set $F(\overline{B_U}(0,1))$ is bounded in V. (Makes sense. The operator norm is basically measuring how big you can make the unit ball.)
Since $Fx = \|x\|F(\frac{x}{\|x\|})$, for all $0 \neq x \in U$, it follows that when $U \neq \{0\}$, we have

$$\|F\| = \sup\{\|Fx\| \mid x \in U, \|x\| = 1\}$$

and that

$$\|Fx\| \le \|F\|\|x\|, \forall x \in U$$

(The equality is because for $x, \|x\| < 1, \|x\|F(\frac{x}{\|x\|}) < 1 * F(\frac{x}{\|x\|})$ which is automatically a vector of norm 1.)
(The inequality is because $\|Fx\| = \|\|x\|F(\frac{x}{\|x\|})\| \le \|x\|\|F\|$)
with

$$\|F\| = \inf\{m \ge 0 \mid \|Fx\| \le m\|x\|, \forall x \in U\}$$

Denote the space of bounded linear operators $F : U \to V$ by $\mathcal{B}(U, V)$.

$$\mathcal{B}(U, V) = \{F : U \to V \mid F \text{ linear}, \|F\| < \infty\}$$

Example 1.11:

Theorem 1.12:
Let U, V be normed linear spaces. Then,

1. The set $\mathcal{B}(U, V)$ is a normed linear space using the operator norm.
2. If V is a Banach space, then $\mathcal{B}(U, V)$ is a Banach space.

Proof:
Part I:
Check it's a subspace:
Let $F, G : U \to V$ be linear operators.

Check Positive definitivesness:
From definition of F, $\|F\| \geq 0$, with $\|F\| = 0 \iff F = 0$.

Check tF is bounded under the operator norm:
For all $x \in U, t \in \mathbb{F}$, we have

$$\|(tF)x\| = \|t(Fx)\| = |t|\|Fx\| \leq |t|\|f\|\|x\|$$

Check sums are bounded under the operator norm:

$$\|(F+G)x\| = \|F(x) + G(x)\| \leq \|F(x)\| + \|G(x)\| \leq \|F\|\|x\| + \|G\|\|x\| = (\|F\| + \|G\|)\|x\|$$

It follows that $\|F + G\| \leq \|F\| + \|G\|$. Thus $B(U, V)$ is indeed a vector space, (It is a subspace of the space HOM(U, V) of linear maps from U to V. Because $0 \in B(U, V)$, and that when F, G bdd, so if tF, and $F + G$). So the operator norm is indeed a norm on $B(U, V)$.

Part II:
Suppose that V is a banach space. Let $(F_n)_{n\geq 1}$ be a cauchy sequence in $B(U, V)$. Note that for each $x \in U, k, l \in \mathbb{Z}^+$, we have

$$\|F_k x - F_\ell x\| = \|(F_k - F_\ell)x\| \leq \|F_k - F_\ell\|\|x\|$$

Now, it follows that $(F_n x)_{n\geq 1}$ is a Cauchy sequence in V (for each x, $\|x\|$ would be fixed. And $\|F_k - F_\ell\|$ is going to 0 in the cauchy sequence sense.
Define a function G based on Cauchy-ness Hence $\|F_k x - F_\ell x\|$ must also obey the same cauchy convergence as it is bounded above by $\|F_k - F_\ell\|\|x\|$), and since that V is a Banach space, the sequence $F_n x$ converges. Define $G : U \to V$ by $Gx = \lim_{n\to\infty} F_n x$. Note that G is linear. (Since F is linear.)
G is bounded.
We now claim that G is a bounded function under the operator norm. (G takes an element in U and maps it to a sequence defined by F_n then the convergence of the sequence.)
Since the sequence F_n is Cauchy in $B(U, V)$, it is also bounded in $B(U, V)$. We pick $M \geq 0$ such that $\|F_n\| \leq M$ for all $n \in \mathbb{Z}^+$. For all $x \in U$, $F_n x \to Gx$ in V, this implies that $\|F_n x\| \to \|Gx\|$ in $\mathbb{R}$. $(|\|F_n x\| - \|Gx\|| \leq \|F_n x - Gx\|)$, and since $\|F_n x\| \leq \|F_n\|\|x\| \leq M\|x\|$ for all n, we have

$$\|Gx\| = \|\lim_{n\to\infty} F_n x\| = \lim_{n\to\infty} \|F_n x\| \leq M\|x\|$$

This shows that $\|G\| \leq M$ so indeed G is bounded.
Claim that $F_n \to G$ in $B(U, V)$ to see that G is proof that F_n converges
Let $\epsilon > 0$. Since F_n is Cauchy in $B(U, V)$, we can pick $M \in \mathbb{Z}^+$ such that for all $k, n \geq m$, $\|F_n - F_k\| < \epsilon$. Then, when $k, n \geq m$, for all $x \in U$, we would have

$$\|F_n x - F_k x\| = \|(F_n - F_k)x\| \leq \|F_n - F_k\|\|x\| < \epsilon\|x\|$$

Letting $n \geq m$, then for all $x \in U$,

$$\|(F_n - G)x\| = \|F_n x - Gx\| = \|F_n x - \lim_{k\to\infty} F_k(x)\| = \lim_{k\to\infty} \|F_n(x) - F_k(x)\| \leq \epsilon\|x\|$$

This shows that $F_n \to G$ in $B(U, V)$ as claimed.

□

Definition 1.11 (Lipschitz continuous): Let $(X, d_X), (Y, x_Y)$ be metric spaces, and $f : X \to Y$. We say that f is Lipschitz continuous on X if there exists constant $\ell \geq 0$, called a Lipschitz constant for f,, where for all $x, y \in X$, we have $d_Y(f(x), f(y)) \leq \ell d_X(x, y)$.

Note that lipshitz continuous implies uniform continuous. Also, if $x_n \to a$ in X then so does $f(x_n) \to f(a)$ in Y. Likewise, (x_n) cauchy in X, then $(f(x_n))$ cauchy in Y.

Theorem 1.13 (Relationship between lipschitz, continuity, and bounded function):
Let U, V be normed linear spaces. Let $F : U \to V$ be a linear map. Then TFAE:

1. F is Lipschitz continuous on U.
2. F is continuous at some point $a \in U$.
3. F is continuous at 0.
4. F is bounded.

In this case, $\|F\|$ is the lipschitz constant for F.

Proof:
Note that the claim F is linear is highly important.
$1 \implies 2$:
If F is lipschiz continuous on U then F is continuous at every point $a \in U$ by the ϵ, δ characterization.
$2 \implies 3$:
Suppose F is continuous at some point $a \in U$. Given $x \in U$ we let $u = x + a$, then $\|x\| = \|u - a\|$ and $\|Fx - F0\| = \|Fx\| = \|Fu - Fa\|$. So as u approach a, x approach 0, so the ϵ, δ definition for continuity works the same here.
$3 \implies 4$:
Suppose that F is continuous at 0. So for we can pick $\delta > 0$ so that $\forall x \in U$, $\|x\| \leq \delta$, we have $\|Fx\| \leq 1$, (This is the definition of δ, ϵ limit where 1 is the ϵ.). So for all $x \in U$ with $\|x\| = 1$, we have $\|\delta x\| = \delta$, so that $\|Fx\| = \frac{1}{\delta}\|F(\delta x)\| \leq \frac{1}{\delta}$.
It follows that $\|F\| \leq \frac{1}{\delta}$.
$4 \implies 1$:
Suppose that F is bounded, for all $x, y \in U$, we have

$$d(Fx, Fy) = \|Fx - Fy\| = \|F(x - y)\| \leq \|F\|\|x - y\| = \|F\|d(x, y)$$

This shows F is Lipschitz continuous with the Lipschitz constant $\|f\|$. □

1.3 Dual Spaces

Definition 1.12 (Dual space):
The (linear) dual space of a vector space U over field $\mathbb{F}$ is the vector space

$$U^{\#} = Hom(U, \mathbb{F}) = \{f : U \to \mathbb{F} \mid f \text{ is linear}\}$$

The (continuous) dual space of a normed linear space U over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, is the normed linear space

$$U^{*} = \mathcal{B}(U, \mathbb{F}) = \{f : U \to \mathbb{F} \mid f \text{ is linear with } \|f\| < \infty\}$$

By previous theorem, U^* is a Banach space.

Theorem 1.14 (The Riesz Representation Theorem for the ℓ_p Spaces):
Let $p, q \in [1, \infty]$ be conjugate indices, i.e. $1/p + 1/q = 1$.
Then

1. The map $F : \ell_q \to \ell_p^*$ given by $F(b)(a) = \sum_{k=1}^{\infty} a_k b_k$ is well defined, linear, injective, and norm-preserving.
2. When $p \neq \infty$, the above map F is also surjective, so we have $\ell_p^* \cong \ell_q$

Proof:
<u>Proof of 1</u>
Think about F eating ℓ_q first then it results in a function that eat ℓ_p. The order does not matter, what matters is it that it takes in one of each.
For $b \in \ell_q$, we write $F(b) = F_b$. For $a \in \ell_p$, we write $|a| = (|a_1|, |a_2| \ldots)$, same with $|b|$. Now, Holder's inequality gives

$$|F_b(a)| = \left|\sum_{k=1}^{\infty} a_k b_k\right| \leq \sum_{k=1}^{\infty} |a_k b_k| = \||a||b|\|_1 \leq \|a\|_p \|b\|_q$$

This means that F_b is a well defined bounded linear map with $F_b : \ell_p \to \mathbb{F}$, with $\|F_b\| \leq \|b\|_q$. Hence, F is a well defined linear map from $F : \ell_q \to \ell_P^*$. We claim that F preserves norm. We will claim that F preserves norm.
The other side of inequality is shown in three cases: (mainly by finding or constructing an element in ℓ_p to prove that this upper bound can be attained.)

- Suppose $p = 1$ (so $q = \infty$.) Let $b \in \ell_\infty$. Let $e_n = (0, \ldots, 0, 1, 0)$ be the nth standard basis vector in $\mathbb{F}^\infty$. So $\|e_n\|_1 = 1$ and $|F_b(e_n)| = |b_n|$. Hence $\|F_b\| \geq |b_n|$. Since $\|F_b\| \geq b_n$ for all $n \in \mathbb{Z}^+$, we have $\|F_b\| \geq \sup\{|b_n| \mid n \in \mathbb{Z}^+\} = \|b\|_\infty$. We know $\|F_b\| \leq \|b\|_\infty$ already. Therefore, since the $\geq, \leq$ side of the inequalities are both true, F is norm preserving as $\|F_b\| = \|b\|_\infty$.
- Suppose that $1 < p < \infty$. Let $b \in \ell_q$. If $b = 0$ then $F_b = 0$. Now assume that $b \neq 0$. Let $m \in \mathbb{Z}^+$ be large enough such that $b_k \neq 0$ for some $k \leq m$. Let $a = (a_1, a_2 \ldots a_m, 0, 0 \ldots) \in \ell_p$, where each $a_k = \frac{|b_k|^q}{b_k}$ when $k \leq m$ and $b_l \neq 0$, and $a_k = 0$ otherwise. Since $|F_b(a)| \leq \|F_b\| \|a\|_p$, and $F_b(a) = \sum_{k=1}^{m} |b_k|^q$, and that

$$\|a\|_p = \left(\sum_{k=1}^{m}|b_k|^{p(q-1)}\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{m}|b_k|^q\right)^{\frac{1}{p}}$$

Now we have

$$\sum_{k=1}^{m}|b_k|^q = |F_b(a)| \leq \|F_n\|\|a\|_p = \|F_b\|\left(\sum_{k=1}^{m}|b_k|^q\right)^{\frac{1}{p}}$$

So hence

$$\|F_b\| \geq \left(\sum_{k=1}^{m}|b_k|^q\right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{m}|b_k|^q\right)^{\frac{1}{q}} \to \|b\|_q \text{ as } m \to \infty$$

It follows that $\|F_b\| \geq \|b\|_q$. Therefore, $\|F(b)\| = \|b\|_q$. Therefore F preserves norm.

- Suppose that $p = \infty$ (and $q = 1$.) Let $b \in \ell_1$. For each $k \in \mathbb{Z}^+$, let $a_k = 1$ if $b_k \geq 0$, and $a_k = -1$ if $b_k = 0$. And let $a = (a_1, a_2, \ldots)$. So $\|a\|_\infty = 1$ and $F_b(a) = \sum_{k=1}^{\infty}|b_k| = \|b\|_1$. So, $\|F_b\| \geq \|b\|_1$. Hence $\|F_b\| = \|b\|_1$ and F preserves the norm.

Proof of 2

Since every norm preserving map is injective, (why is that?), this proves part 1. For part 2, let $1 \leq p < \infty$, let $f \in \ell_p^*$ be arbitrary. Let $b = (f(e_1), f(e_2), f(e_3), \ldots) \in \mathbb{F}^\omega$. e_k being the kth standard basis vector in $\mathbb{F}^\infty$. We claim that $b \in \ell_q$ and that $F_b = f$, which shows that F is surjective.
There are two cases.

- If $p = 1(q = \infty)$. Let $f \in \ell_1^*$, let $b = (f(e_1), f(e_2)\ldots) \in \mathbb{F}^\omega$. For each $k \in \mathbb{Z}^+$, we have $\|e_k\|_1 = 1$, so hence $|b_k| = |f(e_k)| \leq \|f\|$. Since $|b_k| \leq \|f\|$, for all $k \in \mathbb{Z}^+$, we know that $\|b\|_\infty \leq \|f\| < \infty$. So $b \in \ell_\infty$ as required.Now we show $F_b = f$. Let $a \in \ell_p$. For each $m \in \mathbb{Z}^+$, let $x_m = \sum_{k=1}^{m} a_k e_k = (a_1, \ldots, a_m, 0, 0, \ldots) \in \ell_p$. Note that $x_m \to a$ in ℓ_p. Snice f is continuous and linearm

$$f(a) = \lim_{n\to\infty} f(x_m) = \lim_{m\to\infty}\sum_{k=1}^{m} a_k f(e_k) = \lim_{m\to\infty}\sum_{k=1}^{m} a_k b_k = \sum_{k=1}^{\infty} a_k b_k = F_b(a)$$

- Suppose $1 < p < \infty$. We let $f \in \ell_p^*$, let $b = (f(e_1), f(e_2)\ldots) \in \mathbb{F}^\omega$. If $b = 0$ then $b \in \ell_q$. Suppose $b \neq 0$. Now we proceed similarly in case 2 part 1. Pick $m \in \mathbb{Z}^+$ large enough such that $b_k \neq 0$ for some $k \leq m$. Let $a = (a_1, \ldots a_m, 0, 0, \ldots)$ with $a_k = \frac{|b_k|^q}{b_k}$, when $k \leq m, b_k \neq 0$. Now, as above, have $\sum_{k=1}^{m}|b_k|^q = |f(a)| \leq \|f\|\|a\|_p = (\sum_{k=1}^{m}|b_k|^q)^{\frac{1}{p}}$.

  So $\|f\| \geq (\sum_{k=1}^{m}|b_k|^q)^{\frac{1}{q}}$. Take limit $m \to \infty$, $\|b\|_q \leq \|f\| < \infty$, so $b \in \ell_q$. We can show that $F_b = f$ as we did in case 1. Given $a \in \ell_p$, we let $x_m = \sum_{k=1}^{m} a_k e_k \in \ell_p$. Note $x_m \to a$ in ℓ_p, use the fact taht f is continuous and linear, we can again get that $f(a) = F_b(a)$.

- The reason why $p = \infty, q = 1$ does not work, is that the proof break down for $a \in \ell_\infty, x_m = (a_1, \ldots, a_m, 0\ldots)$, as we dont necessarily have $x_m \to a$ in ℓ_p. F is not surjective. (Using the Hahn-Banach Theorem.)

□

Theorem 1.15 (The Riesz Representation Theorem for the L_p spaces):
Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $A \subseteq \mathbb{R}$ be measurable with $\lambda(A) > 0$. Then

- The map $F : L_q(A) \to L_p(A)^*$ given $F(g)(f) = \int_A fg$ is a well-defined injective norm-preserving map.
- When $1 \le p < \infty$, the above map F is also surjective, so we have $L_p(A)^* \cong L_q(A)$.

Proof:
OMITTED. □

1.4 Uniform Boundedness

Definition 1.13: Let X be a metric space, and let $A \subseteq X$. A is dense in X when every nonempty open ball $B \subseteq X$, we have $B \cap A \neq \varnothing$, equivalently, $\overline{A} = X$. A is nowhere dense in X if every nonempty ball $B \subseteq \mathbb{R}$, there exists an nonempty open ball $C \subseteq B$ with $C \cap A = \varnothing$. Eqiuvalently when $\overline{A}^o = \varnothing$. Note that o denotes interior.
When $A \subseteq B \subseteq X$, if A is dense in X so is B, and if B is nowhere dense in X so is A. When $A, B \subseteq X, B = A^c = X \backslash A$, then A nowhere dense $\iff \overline{A}^o = \varnothing \iff \overline{B^o} = X \iff$ interior of B is dense.
If $A \subseteq X$, we say A is first category or meagre if A is countable union of nowhere dense set. A is second category when it is not first category. A is residual if A^C is first category.

Theorem 1.16 (The Baire Category Theorem):
Let X be complete metric spaec.

1. Every first category set in X has empty interior.
2. Every residual set in X is dense
3. Every countable union of closed sets with empty interiors in X has empty interior.
4. Every countable intersection of dense open sets in X is dense.

Proof: 1,2 equivalent by taking complement, 3,4 are special cases of 1,2. Just prove 1.
Let $A \subseteq X$ be first category. Say $A = \bigcup_{n=1}^{\infty} C_n$, where each C_n nowhere dense. Suppose towards contradiction A has nonempty interior. Pick an open ball $B_0 = B(a_0, r_0), r_0 \in (0, 1), \overline{B_0} \subseteq A$.
Since each C_n is nowhere dense, we can pick a nested sequence of open balls $B_n = B(a_n, r_n)$ with $r_n \in (0, 2^{-n})$, $\overline{B_n} \subseteq B_{n-1}$, and $\overline{B_n} \cap C_n = \varnothing$. Since $r_n \to 0$, $\{a_n\}$ is cauchy, and X is complete, get that $\lim_n a_n = a$ converge in X. Note that $a \in \overline{B_n}$ for all n, as $a_k \in \overline{B_n} \forall k \ge n$. Since $a \in \overline{B_0}, \overline{B_0} \subseteq A$, we have $a \in A$. Since $a \in \overline{B_n} \forall n \ge 1$, $\overline{B_n} \cap C_n = \varnothing$. So $a \notin C_n \forall n \ge 1$. So $a \notin \bigcup_{n=1}^{\infty} C_n$. So $a \notin A$. □

Definition 1.14: Following is a notation. Let X be a set. For any set C of subsets of X, we write

$$C_\sigma = \left\{ \bigcup_{k=1}^{\infty} A_k \mid \text{each } A_k \in C \right\} \text{ and } C_\delta = \left\{ \bigcap_{k=1}^{\infty} A_k \mid \text{ each} A_k \in C \right\}$$

Note $C_{\sigma\sigma} = C_\sigma, C_{\delta\delta} = C_\delta$.

Definition 1.15: Let X be a set. A σ algebra in X is a set C of subsets of X such that

1. $\varnothing \in C$
2. If $A \in C$, then $A^C \in C$
3. If $A_1 \ldots \in C$, then $\bigcup_{k=1}^{\infty} \in C$.

We have $C_\sigma = C$, and $C_\delta = C$.

In a metric or a topological space X, we let $\mathcal{G}$ denote the set of all open sets in X, let F denote the set of all closed subsets in X we have $\mathcal{G}_\delta = \mathcal{G}$ and that $F_\delta = F$.
Using BCT, we show in $\mathbb{R}$ we have $F \subseteq \mathcal{G}_\delta$, $\mathcal{G} \subseteq F_\sigma$. $F_\sigma \neq \mathcal{G}_\delta$. And $\mathcal{G}_\delta \cup F_\sigma \subsetneq \mathcal{G}_{\delta\sigma} \cup F_{\sigma\delta}$.

Theorem 1.17 (The Banach-Steinhaus Theorem, Uniform Boundedness Principal):
Let X be a Banach space. Y be a normed linear space. Let A be a set of bounded linear maps $L : X \to Y$. Suppose that every $x \in X$, there exists $m_x \geq 0$ such that $\|Lx\| \leq m_x$ for all $L \in S$, then there exists $m \geq 0$ such that $\|L\| \leq m, \forall L \in S$.

Proof: OMITTED □

Theorem 1.18 (Condensation of Singularities):
Let X be a Banach space, let Y be a normed linear space. For each $m, n \in \mathbb{Z}^+$, we let $L_{m,n} : X \to Y$ be a bounded linear map. Suppose that for each $m \in \mathbb{Z}^+$, there exists $x_m \in X$ such that

$$\limsup_{n\to\infty} \|L_{mn}(x_m)\| = \infty.$$

Then the set

$$E = \left\{ x \in X \mid \limsup_{n\to\infty} \|L_{mn}(x)\| = \infty \text{ for all } x \in \mathbb{Z}^+ \right\}$$

Is a dense $\mathcal{G}_\delta$ set.

Proof: OMITTED □

2 Section 1. Hilbert Spaces

2.1 Review

Definition 2.1:

- Hamel basis

- Hamel dimension
- Orthonormal
- Orthogonal

Theorem 2.1:
Let V be a ips. Let $B \subset V$ be orthogonal. Let $x, y \in Span(B)$ with $x = \sum_{k=1}^{n} a_k u_k, y = \sum_{k=1}^{n} b_k u_k$, $a_k, b_k \in \mathbb{F}$, $u_k \in \mathcal{B}$.
Then,

$$\langle x, u_k \rangle = a_k, \langle x, y \rangle = \sum_{k=1}^{n} a_k \overline{b_k}, \|x\|^2 = \sum_{k=1}^{n} |a_k|^2$$

particularly B is linearly independent.

Theorem 2.2 (Theorem and corollaries of this theorem):

1. Gram Schmidt procedure: give you V, a ips, which is finite or countable hamel dimension. If $A = (u_1, \ldots)$ is a dinite or countable ordered hamel basis of V, then we can make a orthogonal hamel basis by the gram schmidt process.
2. Every ips with finite or countable hamel dimension has a orthonormal hamel basis.
3. Let V be an ips with finite or countable hamel dimension. Let $U \subset V$ be finite dimensional subspace. Then every orthogonal hamel basis for U extend to an orthogonal hamel basis for V.
4. U, V are ips of finite or countable hamel dimension. Then U, V are isomorphic if and only if they have same dimension. If $\dim(U) = n$, then U is isomorphic to $\mathbb{F}^n$. If $\dim(U) = \aleph_0$ then U is isomorphic to $\mathbb{F}^\infty$. This is the space of sequences in $\mathbb{F}$ with only finitely many nonzero terms, using inner product $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$.
5. Every finite dimensional ips is complete, and every ips with countable hamel dimension is not complete.

Definition 2.2: When W is a vector space and $U, V \subseteq W$ are subspaces, we write $U + V = \{u + v \mid u \in U, v \in V\}$, and write $W = U \bigoplus V$ when $W = U + V$ and that $V \cap U = \{0\}$. That is, every $x \in W, x = u + v$ for u, v unique.

Definition 2.3: Let V be ips. For subspace $U \subseteq V$, define orthogonal complement U in V to be

$$U^\perp = \{x \in V \mid \langle x, u \rangle = 0, \forall u \in U\}$$

Theorem 2.3:
Let V be ips, $U \subseteq V$ be a subspace. Then

1. $U^{\perp}$ is subspace of V.
2. If $\mathcal{B}$ is a bsis for U, then $U^{\perp} = \{x \in V \mid \langle x, u \rangle = 0, \forall u \in \mathcal{B}\}$
3. $U \cap U^{\perp} = \{0\}$
4. $U \subseteq (U^{\perp})^{\perp}$

   If U is finite dimensional, then we also have

   - $U \bigoplus U^{\perp} = V$
   - $U = (U^{\perp})^{\perp}$

Definition 2.4: Let V be ips. Let $U \subseteq V$ be subspace such that $V = U \bigoplus U^{\perp}$. For $x \in V$, we define orthogonal projection of x onto U, denoted by $\text{proj}_u(x)$ as follows. Since $V = U \bigoplus U^{\perp}$, we can pick unique vectors $u, v \in V$ with $u \in U, v \in V, u + v = x$. Then define

$$\text{proj}_U(x) = u$$

When U is finite dimensional, $U = (U^{\perp})^{\perp}$, for u, v as above, have $\text{proj}_{U^{\perp}}(x) = v$. When $y \in V, U = \text{Span}\, y$, we also have $\text{proj}_y(x) = \text{proj}_U(x)$.

Definition 2.5: Let V be ips. $U \subseteq V$ be subspce of V such that $U \bigoplus U^{\perp}$. Let $x \in V$, then $\text{proj}_U(x)$ is the unique point in U that is nearest to x.

Example 1 V be ips. U be finite diml subspace of V. $B = \{u_1, \ldots u_n\}$ be orthonormal basis for U. Recall

$$\text{proj}_u(x) = \sum_{k=1}^{n} \frac{\langle x, u_1 \rangle}{\|u_k\|^2} u_k$$

□

2.2 Closed subspaces of Hilbert Spaces and Orthogonal Projections

Definition 2.6: V be vector space over $\mathbb{F}$. For subset $S \subseteq V$, S is convex when $\forall a, b \in S, a + t(b - a) \in S, \forall t \in (0, 1)$.

Theorem 2.4:
H be Hilbert space. Let $\varnothing \neq S \subseteq H$ be closed, convex. Then for every $a \in H$, there exists unique point $b \in S$ that is nearest to a. I.e. $\|a - b\| \leq \|a - x\|, \forall x \in S$.

Proof:
Show the existence of the smallest
Let $a \in H$. Let $d = \text{dist}(a, S) = \inf\{\|x - a\| \mid x \in S\}$. Choose a sequence $\{x_n\}$ in S such that $\|x_n - a\| \to d$,

so $\|x_n - a\|^2 \to d^2$. Let $\epsilon > 0$, we pick $M \in \mathbb{Z}^+$ so that for all $n \geq m$, $\|x_n - a\|^2 \leq d^2 + \frac{\epsilon^2}{4}$. Let $k, l \geq m$, by the parallelogram law, we have

$$\|(x_k - a) + (x_l - a)\|^2 + \|(x_k - a) - (x_l - a)\|^2 = 2\|x_k - a\|^2 + 2\|x_l - a\|^2$$

S is convex, so $\frac{x_k + x_l}{2} \in S$, so $\|\frac{x_k+x_l}{2} - a\| \geq d$. So that

$$\begin{aligned}\|x_k - x_l\|^2 &= \|(x_k - a) - (x_l - a)\|^2 \\ &= 2\|x_k - a\|^2 + 2\|x_l - a\|^2 - \|(x_k - a) + (x_l - a)\|^2 \\ &= 2\|x_k - a\|^2 + 2\|x_l - a\|^2 - 4\left\|\frac{x_k + x_l}{2} - a\right\|^2 \\ &\leq 2(d^2 + \frac{\epsilon^2}{4}) + 2(d^2 + \frac{\epsilon^2}{4}) - 4d^2 = \epsilon^2\end{aligned}$$

So $\|x_k - x_l\| \leq \epsilon$. This shows that the sequence $\{x_n\}$ is cauchy. Since H is complete, $\{x_n\}$ converges in H, since S is closed in H, the limit is in S. Let $\lim_{n\to\infty} x_n = b \in S$. Since $b \in S, \|d - a\| \geq d$, and we have $\|b - a\| \leq \|b - x_n\| + \|x_n - a\|$ for all $N \in \mathbb{Z}^+$. $\|b - a\| \leq \lim_{n\to\infty}(\|b - x_n\| + \|x_n - a\|) = 0 + d$. So that $\|b - a\| = d$. Hence $\|d - a\| \geq \|x - a\|$ for all $x \in S$.

Show uniqueness

Finally, note that b is unique because given $c \in S, \|c - a\| = d$, since S is convex, have $\frac{b+c}{2} \in S$, so $\left\|\frac{b+c}{2} - a\right\| \geq d$, parallelogram shows

$$\|b - c\|^2 = \|(b - a) - (c - a)\|^2 = 2\|b - a\|^2 + 2\|c - a\|^2 - \|(b - a) + (c - a)\|$$

$$= 4d^2 - 4\left\|\frac{b - c}{2} - a\right\|^2 \leq 4d^2 - 4d^2 = 0$$

So $\|b - c\| = 0 \implies b = c$.

□

Theorem 2.5:
Let H be a Hilbert space. Let $U \subseteq H$ be a subspace. Then U is closed if and only if $H = U \bigoplus U^\perp$. In this case, $U^\perp$ is closed and $(U^\perp)^\perp = U$. For $x \in H$, if $x = u + v$ with $u \in U, v \in U^\perp$, then u is unique point in U nearest x, and v is the unique point in $U^\perp$ nearest x.

Proof: Proof omitted Proof confused □

Definition 2.7: If H is a hilbert space, $U \subseteq H$ is closed subspace, we define the orthogonal projection onto U to be the map $P : H \to U$, given by $Px = u$ where u is the unique point in U nearest to x. Equivalently, $Px = u$ where $x = u + v$ with $u \in U, v \in U^\perp$.

2.3 Unordered series

Definition 2.8: Let $(a_n)_{n\geq 1}$ be a sequence in a normed linear space V. We say $\sum_{k=1}^{\infty}$ converges absolutely in V when $\sum_{k=1}^{\infty}\|a_k\|$ converges in $\mathbb{R}$.
$\sum_{k=1}^{\infty} a_k$ converges unconditionally in V when $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges in V for every bijective map $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$.

Note that sequences in $\mathbb{R}$, the series $\sum|a_n|$ converges if and only if $\sum|a_{\sigma n}|$ converges for every bijective map σ between positive integers. In $\mathbb{R}$, unconditional convergence is absolute convergence. In ℓ_2, the series $\sum \frac{1}{n}e_n$ converges unconditionally but does not converge absolutely.

Definition 2.9: Let K be nonempty set, possible uncountable. When A is any set, an indexed set in A with index set K is a function $a : K \to A$, and we write $a_k = a(k)$, and $a = (a_k)_{k\in K}$. When X is a normed linear space and $(a_k)_{k\in K}$ is an indexed set in X, the unordered series $\sum_{k\in K} a_k$ (Note how it is $k \in K$, without an order) is defined to be the indexed set $(s_f)_{f\in \text{Fin}(K)}$, where $\text{Fin}(K)$ is the set of finite subsets of K and $s_F = \sum_{k\in F} a_k$ for each $F \in \text{Fin}(K)$. (Cuz finite sets are summable, so we characterize the possibly uncountable set K with its finite subsets, and see how sums behave under those finite subsets). We say that the unordered series $\sum_{k\in K} a_k$ converges (unconditionally) in X when there exists $s \in X$ such that

$$\forall \epsilon > 0, \exists F \in \text{Fin}(K), \forall I \in \text{Fin}(K), (I \supseteq F \implies \|s_I - s\| < \epsilon)$$

(It says the sum converges if we can find a finite subset and sum up those subsets so that the subset sum can approach arbitrarily close to the real sum).
In this case, $s \in X$ is unique, and it is called the unordered sum of the unordered series $\sum_{k\in K} a_k = s$. As usual, we write $\sum_{k\in K}$ both to denote the unordered series (may or may not converge) and its sum (when it converges).
We say the unordered series $\sum_{k\in K} a_k$ converges absolutely in X when $\sum_{k\in K}\|a_k\|$ converges in $\mathbb{R}$.
When $(a_k)_{k\in K}$ is an indexed set in $\mathbb{R}$ with each $a_k \geq 0$, whether or not the unordered series $\sum_{k\in K} a_k$ converges, we define its unordered sum to be

$$s = \sup\{\sum_{k\in F} a_k \mid F \in \text{Fin}(K)\}$$

Write $\sum_{k\in K} a_k = s$. Should verify as an exercise that unordered series converges if and only if its sum is finite and that in this case, our two definitions of the sum agree.

Theorem 2.6:
Let $(a_k)_{k\in K}$ be an indexed set in $\mathbb{R}$ with each $a_k \geq 0$. If $\sum_{k\in K} a_k$ converges then there are at most countably many indices $k \in K$ for which $a_k \neq 0$.

Proof: For each $n \in S^+$, we let $K_n = \{k \in K \mid a_k \geq \frac{1}{n}\}$. So, if any of the sets K_n is infinite, we would have $\sum_{k\in K} a_k = \infty$. So, if $\sum_{k\in K} a_k < \infty$, every set K_n must be finite. So the set

$$\{k \in K \mid a_k > 0\} = \bigcup_{n=1}^{\infty} K_n$$

is at most countable. □

Definition 2.10:
Let $(a_k)_{k\in K}$ be an indexed set in a normed linear space X. We say that the unordered series $\sum_{k\in K} a_k$ is cauchy when

$$\forall\epsilon > 0, \exists F \in \text{Fin}(K), \forall I, J \in \text{Fin}(K), (I, J \supseteq F \implies \|s_I - s_J\| < \epsilon)$$

Now, we can verify as an exercise that $\sum_{k\in K} a_k$ is cauchy if and only if

$$\forall\epsilon > 0, \exists F \in \text{Fin}(K), \forall L \in \text{Fin}(K), (L \cap F = \varnothing \implies \|s_L\| < \epsilon)$$

Theorem 2.7 (Cauchy Criterion for unordered series):
Let $(a_k)_{k\in K}$ be an indexed set in normed linear space X.

1. If $\sum_{k\in K} a_k$ converges in X then it is cauchy.
2. If X complete and $\sum_{k\in K} a_K$ is Cauchy, then $\sum_{k\in K} a_k$ converges in X.

Proof:
To prove part 1, suppose that $\sum_{k\in K}$ converges in X. Say $s = \sum_{k\in K} a_k$. Let $\epsilon > 0$. Choose $F \in \text{Fin}(K)$ such that all $I \in \text{Fin}(K)$ with $I \supseteq F$ we have $\|s_I - s\| < \epsilon/2$. Let $I, J \in \text{Fin}(K)$ with $I, J \supseteq F$. Then we have $\|s_I - s_J\| \leq \|s_I - s\| + \|s - s_J\| < \epsilon$, so that $\sum_{k\in K} a_K$ is cauchy.
Prove part 2. Suppose that X is Banach space and that $\sum_{k\in K} a_k$ is Cauchy in X. Let $\epsilon > 0$. Since $\sum_{k\in K} a_k$ is cauchy, we pick sets $F_n \in \text{Fin}(K)$ with $F_1 \subseteq F_2 \subseteq \ldots$ such that all $I, J \in \text{Fin}(K)$ such that $I, J \supseteq F_n$, we have $\|s_I - s_J\| < \frac{\epsilon}{2^b}$. (we can achieve this by having chosen F_n, then pick $G_n \in \text{Fin}(K)$ so $G_n \subseteq I, J \in \text{Fin}(K)$ implies $\|s_I - s_J\| < \frac{1}{2^{n+1}}$, set $F_{n+1} = F_n \cup G_n$).
Then the sequence $(s_{F_n})_{n\geq 1}$ is cauchy in X. (when $l > m$, we have $\|s_{F_l} - s_{F_m}\| \leq \|s_{F_l} - s_{F_{l-1}}\| + \ldots + \|s_{F_{m+1}} - s_{F_m}\| < \frac{1}{2^{l-1}} + \ldots + \frac{1}{2^m} < \frac{1}{2^{m-1}}$.)
Now, X is complete, so the sequence $(s_{F_n})_{n\geq 1}$ converges. Say $(s_{F_n}) \to s$ in X. We claim that $\sum_{k\in K} a_k = s$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ with $\frac{1}{2^m} < \epsilon/2$, such that $n \geq m$ implies $\|s_{F_n} - s\| < \frac{\epsilon}{2}$. Then for all $I \in \text{Fin}(K)$ with $I \supseteq F_m$, we would have $\|s_I - s\| \leq \|s_I - s_{F_m}\| + \|s_{F_m} - s\| < \frac{1}{2^M} + \epsilon/2 < \epsilon$. Therefore, $\sum_{k\in K} a_k = s$ as claimed.

□

2.4 Formulas Involving Orthonormal Indexed sets

Definition 2.11: An indexed set $(u_k)_{k\in K}$ in an inner product space V is called orthonormal when $\|u_k\| = 1, \forall k \in K$ and $\langle u_k, u_\ell\rangle = 0, \forall k, \ell \in K, k \neq \ell$.

Theorem 2.8:
Let H be a hilbert space. Let $(u_k)_{k\in K}$ be an orthonormal indexed set in H, and let $\mathcal{B} = \{u_k \mid k \in K\}$. Let $x, y \in \overline{\text{Span}\,\mathcal{B}} \in H$. For each $k \in K$, let $a_k = \langle x, u_k\rangle$ and $b_k = \langle y, u_k\rangle$. Then

1. $\sum_{k\in K} a_k u_k = x$
2. $\sum_{k\in K} |a_k|^2 = \|x\|^2$
3. $\sum_{k\in K} a_k \overline{b_k} = \langle x, y\rangle$

Theorem 2.9:
Let H be a hilbert space. Let $(u_k)_{k\in K}$ be an orthonormal indexed set in H. Let $\mathcal{B} = \{u_k \mid k \in K\}$, let $U = \overline{\text{Span}\,B}$. Let $(c_k)_{k\in K}$ be an indexed set in $\mathbb{F}$. Then

1. If $\sum_{k\in K} c_k u_k$ converges in H and $x = \sum_{k\in K} c_k a_k$, then $x \in U$ and $c_k = \langle x, u_k\rangle$ for all $k \in K$.
2. $\sum_{k\in K} c_k u_k$ converges in H if and only if $\sum_{k\in K} |c_k|^2$ converges in $\mathbb{R}$.

Theorem 2.10 (Bessel's Inequality):
Let V be an inner product space. Let $(u_k)_{k\in K}$ be an orthonormal indexed set in V. For all $x \in V$, we have $\sum_{k\in K} |\langle x, u_k\rangle|^2 \leq \|x\|^2$.

Proof:
Let $x \in V$. For $k \in K$, let $a_k = \langle x, u_k\rangle$. Let $F \in \text{Fin}(K)$ and let $w_F = \sum_{k\in F} a_k u_k$.
So

$$\begin{aligned}
0 &\leq \|x - w_F\|^2 \\
&= \|x\|^2 - 2Re\langle x, w_F\rangle + \|w_F\|^2 \\
&= \|x\|^2 - 2Re\left(\sum_{k\in F} \overline{a_k}\langle x, u_k\rangle\right) + \sum_{k,l\in F} a_k \overline{a_l}\langle u_k, u_l\rangle \\
&= \|x\|^2 - \sum_{k\in F} |a_k|^2
\end{aligned}$$

This means that $\sum_{k\in F} |a_k|^2 \leq \|x\|^2$ for every $F \in \text{Fin}(K)$. It follows that $\sum_{k\in K} \|a_k\|^2 \leq \|x\|^2$, as required.

□

Theorem 2.11 (Orthogonal Projection):
Let H be a Hilbert space. Let $(u_k)_{k\in K}$ be an orthonormal indexed set in H. Let $\mathcal{B} = \{u_k \mid k \in K\}$, let $U = \overline{\text{Span}\,\mathcal{B}}$. The orthogonal projection $P : H \to U$ is given by $Px = \sum_{k\in K} a_k u_k$ where $a_k = \langle x, u_k\rangle$, and we have $\|P\| = 1$.

2.5 Hilbert Bases

Theorem 2.12:
Let H be a hilbert space, and let $\mathcal{B}$ be an orthonormal set in H. Then, $\mathcal{B}$ is a maximal orthonormal set if and only if $\operatorname{Span}\mathcal{B}$ is dense in H.

Theorem 2.13:

1. Every inner product space contains a maximal orthonormal set
2. In a Hilbert space, any two maximal orthonormal sets have the same cardinality

Definition 2.12 (Hilbert basis): Note that a Hilbert basis is defined differently than a Hamel basis. A Hilbert basis for a hilber space H is a maximal orthonormal set in H. The Hilbert dimension of a Hilbert space H, denoted by $\dim H$, is the cardinality of any Hilbert basis for H. Now we default on the Hilbert dimension instead of the Hamel dimension.

Theorem 2.14:
Let H be a Hilbert space. Let $(u_k)_{k\in K}$ be an orthonormal indexed set in H. Let $\mathcal{B} = \{u_k \mid k \in K\}$. THen the following are equivalent.

1. $\mathcal{B}$ is a Hilbert basis for H.
2. For every $x \in H$, we have $x = \sum_{k\in K} a_k u_k$, where $a_k = \langle x, u_k\rangle$.
3. For every $x \in H$, we have $\|x\|^2 = \sum_{k\in K} |a_k|^2$, where $a_k = \langle x, u_k\rangle$.
4. For every $x \in H$, we have $\langle x, y\rangle = \sum_{k\in K} a_k \overline{b_k}$, where $a_k = \langle x, u_k\rangle$ and $b_k = \langle y, u_k\rangle$.

Theorem 2.15:
Let H be a hilbert space with hilbert basis $\mathcal{B}$. Then H is separable if and only if $\mathcal{B}$ is at most countable.

Proof:
$\implies$ for contrapositive
Suppose $\mathcal{B}$ is uncountable. Let S be any dense subset of H. Goal is to show that S is uncountable so that H is not separable. For each $u \in \mathcal{B}$, choose $s_u \in S$ with $\|s_u - u\| \leq \frac{\sqrt{2}}{4}$. For $u, v \in \mathcal{B}$ with $u \neq v$, we have $\|u\| = 1$, $\|v\| = 1$ and $\langle u, v\rangle = 0$, such that $\|u - v\|^2 = \|u\|^2 + \|v\|^2 = 2$. So

$$\|s_u - s_v\| = \|(s_u - u) + (u - v) + (v - s_v)\| \geq \|u - v\| - (\|s_u - u\| + \|s_v - v\|) = \sqrt{2} - \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} > 0$$

This means that $s_u \neq s_v$. This means that S is uncountable.
$\impliedby$
Suppose that $\mathcal{B} = \{u_1, \ldots\}$ is finite or countable. The previous theorem states that $\operatorname{Span}_{\mathbb{F}}\mathcal{B}$ is dense in H. Note that $\operatorname{Span}_{\mathbb{Q}}\mathcal{B}$ is dense in $\operatorname{Span}_{\mathbb{R}}\mathcal{B}$ and that $\operatorname{Span}_{\mathbb{Q}[i]}\mathcal{B}$ is dense in $\operatorname{Span}_{\mathbb{C}}\mathcal{B}$. So, given $c_1 \ldots, c_n \in \mathbb{F}$, we

can pick $r_1, \dots r_n \in \mathbb{K}$, where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{Q}[i]$, such that $|r_k - c_k| < \frac{\epsilon}{n}$, then

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} r_k u_k - \sum_{k=1}^{n} c_k u_k \right\| \\
=&\left\| \sum_{k=1}^{n} (r_k - c_k) u_k \right\| \\
\leq&\sum_{k=1}^{n} \|(r_k - c_k) u_k\| \\
=&\sum_{k=1}^{n} |r_k - c_k| \|u_k\| \\
=&\sum_{k=1}^{n} |r_k - c_k| < \epsilon
\end{aligned}$$

□

2.6 The dual space and the adjoint map

Theorem 2.16 (The Riesz Representation Theorem for Hilbert Spaces):
Let H be a Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The map $\phi : H \to H^*$, given by $\phi(u)(x) = \langle x, u \rangle$ is a bijective norm preserving map which is linear when $\mathbb{F} = \mathbb{R}$ and conjugate linear when $\mathbb{F} = \mathbb{C}$.

Proof:
For $u \in H$, we write $\phi_u = \phi(u)$ such that $\phi_u(x) = \phi(u)(x) = \langle x, u \rangle$.
To show it is norm preserving:
We know $\phi_u(u) = \langle u, u \rangle = \|u\|^2$, we know that $\|\phi_u\| \geq \|u\|$. Also for all $x \in H$, $|\phi_u(x)| = |\langle x, u \rangle| \leq \|x\| \|u\|$, we have $\|\phi_u\| \leq \|u\|$. So ϕ_u is a bounded linear map $\phi_u : H \to \mathbb{F}$ with $\|\phi_u\| = \|u\|$. So ϕ is a norm preserving map $\phi : H \to H^*$. ϕ is linear when $\mathbb{F} = \mathbb{R}$ and it is conjugate linear when $\mathbb{F} = \mathbb{C}$.
To show it is bijective
It is injective because it's norm preserving. We now want to show it is surjective. Let $f \in H^*$, so $f : H \to \mathbb{F}$ is a bounded linear map. If $f = 0$, then we can take $u = 0$, so $\phi_u = f$. Now, suppose that $f \neq 0$. We study its kernel. Let $U = \ker(f)$. Since f linear, U is a subspace of H. Since f is bounded (hence continuous), U is closed, from a previous theorem that $H = U \bigoplus U^\perp$. Now, since $f \neq 0$, $U \neq H$, so we have that $U^\perp \neq \{0\}$. We now pick $w \in H$ with $f(w) \neq 0$. We pick $v \in U^\perp$ with $\|v\| = 1$. Let $x \in H$. For $y = f(x)v - f(v)x$, we have $f(y) = f(x)f(v) - f(v)f(x) = 0$, so $y \in \ker(f) = U$. Since $y \in U, v \in U^\perp$, we have $\langle y, v \rangle = 0$. So for all $x \in H$,

$$\begin{aligned}
f(x) = f(x)\|v\|^2 = f(x)\langle v, v \rangle = \langle f(x)v, v \rangle = \langle y + f(v)x, v \rangle \\
= \langle f(v)x, v \rangle = f(v)\langle x, v \rangle = \langle x, \overline{f(v)}v \rangle
\end{aligned}$$

So we can choose that $u = \overline{f(v)}v$ to get $\phi(u) = \phi_u = f$.
The trick is to pick $v \in U^\perp$ with norm 1, so making $u = \overline{f(v)}v$ will give you the item in H that gives you the surjection.

□

Definition 2.13:
When H is a Hilbert space, we use the bijection ϕ of the above theorem to define an inner product on H^* which makes the map $\phi : H \to H^*$ an isomorphism of inner product spaces, as follows. Given $f, g \in H^*$, we choose $u, v \in H$ such that $\phi(u) = f$ and $\phi(v) = g$, (we let u, v be the elements such that $f(x) = \langle x, u\rangle$ and $g(x) = \langle x, v\rangle$. they can be obtained by simply taking the inverse ϕ), then define $\langle f, g\rangle = \langle u, v\rangle$.

Definition 2.14: Recall that when U, V are vector spaces, and $F : U \to V$ is a linear map, we write $U^\#$, $V^\#$ to denote the algebraic dual spaces. We define the dual, (or the transpose, or the algebraic adjoint) of F to be the linear map $F^T : V^\# \to U^\#$ given by $F^T(g) = g \circ F$.
By $F^T(g)(u) = g(F(u))$ when $g \in V^\#$ and $u \in U$. In this case, U, V are normed linear spaces and $F \in \mathcal{B}(U, V)$.
When H, K are Hilbert spaces, over $\mathbb{F}$, $F : H \to K$ is a continuous linear map, we defined the adjoint of F to be the linear $F^* : K \to H$ given by $F^* = \phi^{-1} \circ F^T \circ \psi$, where $\phi : H \to H^*$ is the bijective map given by $\phi(u)(x) = \langle x, u\rangle$, and $\psi : K \to K^*$ is the bijective map given by $\psi(v)(y) = \langle y, v\rangle$. Equivalently, the adjoint of D is the map $F^* : K \to H$ such that $\phi \circ F^* = F^T \circ \psi$, that is the map such that

$$\phi(F^*(y)) = F^T(\psi(y)) = \psi(y) \circ F, \forall u \in H$$

$$\phi(F^*y)(x) = \psi(y)(Fx), \forall x, y \in H$$

$$\langle x, F^*y\rangle = \langle Fx, y\rangle, \forall x, y \in H$$

2.7 Weak Convergence

Definition 2.15 (Converge Weakly): Let V be an ips over F, let (u_n) be a sequence in V and let $w \in V$. We say that (u_n) converges weakly to w in V, write $u_n \to w$ in V when $\langle u_n, x\rangle \to \langle w, x\rangle$ in F for all $x \in V$.

Note that convergence implies weak convergence, but converse is not true. For example, orthonormal sequence converges weakly to 0 in H but does not converge to 0 in H.

Theorem 2.17:
Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

Let H be a hilbert space.
Step 1. Preliminary claim

We first claim that for all $a \in H$, and every bounded sequences $u = (u_n)$ in H, there is a subsequence (u_{n_l}) of (u_n) such that the sequence $(\langle u_{n_l}, a\rangle)$ converges in F. To show this is true, we first let $a \in H$ and $(u_n)_{n\geq 1}$ be a bounded sequence in H. We let $M = \sup\{\|u_n\| \mid n \in \mathbb{Z}^+\}$. Now, $\forall n \in \mathbb{Z}^+$, $|\langle u_n, a\rangle| \leq \|u_n\|\|a\| \leq M\|a\|$. Therefore, the sequence $(\langle u_n, a\rangle)$ is bounded in $\mathbb{F}$. Now by using Bolzano Weierstrauss Theore,, we can pick a subsequence (u_{n_l}) of (u_n) such that $(\langle u_n, a\rangle)$ converges in $\mathbb{F}$ as claimed.

Step 2. For H that is separable

Now suppose that H is separable. We let $S = \{a_1, \ldots\} \subseteq H$ be a countably dense subset. Let $u = (u_n)$ be a bounded sequence in H. Let $M = \sup\{\|u_n\| \mid n \in \mathbb{Z}^+\}$.

- By the claim above, we can pick a subsequence (u_{n_l}) of (u_n) such that $\lim_{l\to\infty}\langle u_{n_l}, a_1\rangle$ exists in $\mathbb{F}$.
- Then we can pick a subsequence $(u_{n_{l_k}})$ of (u_{n_l}) such that $\lim_{k\to\infty}\langle u_{n_{l_k}}, a_2\rangle$ exists in $\mathbb{F}$.
- Then we can pick a subsequence $(u_{n_{l_{k_j}}})$ of $(u_{n_{l_k}})$ such that $\lim_{j\to\infty}\langle u_{n_{l_{k_j}}}, a_3\rangle$ exists in $\mathbb{F}$.
- and so on.

The diagonal sequence $v = (v_1, v_2, v_3, \ldots) = (u_{n_1}, u_{n_{l_2}}, u_{n_{l_{k_3}}}, \ldots)$, is then a subsequence of the original sequence (u_n) such that $(\langle v_k, a_m\rangle)$ converges for every $M \in \mathbb{Z}^+$. That is, $(\langle v_k, a\rangle)$ converges for every $a \in S$.

Now, since we have it on a dense set, we can use this property to extend this to all of H. Being uniformly continuous on a dense set will extend uniquely to a map on H.

Define $f : S \to \mathbb{F}$ by $f(a) = \lim_{k\to\infty}\langle v_k, a\rangle$ for $a \in S$. f is uniformly continuous on S because $\forall a, b \in S$, we have

$$|\langle v_k, a - b\rangle| \le \|v_k\|\|a - b\| \le M\|a - b\|$$

for all k, so that

$$|f(a) - f(b)| = \left|\lim_{k\to\infty}\langle v_k, a\rangle - \lim_{k\to\infty}\langle v_k, b\rangle\right| = \lim_{k\to\infty}|\langle v_k, a - b\rangle| \le M\|a - b\|$$

Since $f : S \to F$ is uniformly continuous in S, and S is dense in H, it follows that f extends uniquely to a continuous map $f : H \to \mathbb{F}$ defined by $f(x) = \lim_{n\to\infty} f(a_n)$ where $x \in H$ and that (a_n) is any sequence in S such that $a_n \to x$ in H. (Verify that f is linear and bounded so $f \in H^*$) with $\|f\| \le M$. Too lazy to verify, rip.)
Now by the Riesz Representation Theorem (by ϕ is surjective), we can pick $w \in H$ such that $f(x) = \langle x, w\rangle$, $\forall x \in H$. Verify that we have $\lim_{k\to\infty}\langle v_k, x\rangle = \langle w, x\rangle$ for all $x \in H$. Hence v_k converges weakly to w in H. This completes the proof for when f is separable.

Step 3. For H that is not separable, use step 2

Now, suppose that H is not separable. Let (u_n) be a bounded sequence in H. Let $\mathcal{B} = \{e_k \mid k \in K\}$ be a Hilbert basis for H. For each $N \in \mathbb{Z}^+$, by a previous theorem, we have $u_n = \sum_{k\in K} c_{n,k}e_k$ where $c_{n,k} = \langle u_n, e_k\rangle$, and that we have $\sum_{k\in K}|c_{n,k}|^2 = \|u_k\|^2$. By another theorem, for each $N \in \mathbb{Z}^+$, there are at most countably many indices $k \in K$, for which $c_{n,k} \neq 0$. Thus the set $L = \{k \in K \mid \exists n \in \mathbb{Z}^+, c_{n,k} \neq 0\}$, is at most countable, and all elements u_n lie in the separable Hilbert space $U = \overline{\text{Span}\{e_l \mid l \in L\}}$. (Use the countable property to extract a countable subset of basis elements from the Hilbert basis to achieve the countable span!.)
Since (u_n) is bounded, as prove above, we can find a subsequence of (u_n) which converges weakly to an element in $w \in U$.

2.8 The Spectral Theorem for Compact Self-Adjoint Operators

Definition 2.16:
Let H be a Hilbert space. A compact operator on H is linear map $F : H \to H$ which sends weakly convergent sequences to convergent sequences, that is a linear map such that if $u_n \to w$ weakly in H then $Fu_n \to Fw$ in H.

Note that when H is a Hilbert space, every compact operator on H is continuous, because $(u_n \to w)$ in H, then $u_n \to w$ weakly in H. But the converse is not always true. If H is infinite diml Hilbert space, then the identity map $I : H \to H$ is continuous but not compact. (i.e. if (u_n) is an orthonormal sequence in H, the $u_n \to 0$ weakly in H but $u_n \not\to 0$ in H).

Definition 2.17:
Let H be a Hilbert space. A self-adjoint operator on H is a continuous linear map $F : H \to H$ such that $F^* = F$, that is such that $\langle Fx, y\rangle = \langle x, F^*y\rangle, \forall x, y \in H$.

Note that if λ is an eigenvalue of F corresponding to the vector x, we would have $Fu = \lambda u$.

Theorem 2.18:
Let H be a Hilbert space and let $F : H \to H$ be a continuous self-adjoint oeprator. Then

1. For every $u \in H$, we have $\langle Fu, u\rangle \in \mathbb{R}$. In particular, every eigenvalue of F is real.
2. We have $\|F\| = \sup\{|\langle Fu, u\rangle| \mid u \in H, \|u\| = 1\}$. In particular, for every every eigenvalue λ of F, we have $|\lambda| \leq \|F\|$.

Proof:
Part i
Since F is self adjoint, $\langle Fu, u\rangle = \langle u, F^*u\rangle = \langle u, Fu\rangle = \overline{\langle Fu, u\rangle}$. So $\langle Fu, u\rangle \in \mathbb{R}$. Now, if λ is an eigenvalue of F, and if $u \in H$ is a vector corresponding the λ, with $\|u\| = 1$, we have $\lambda = \lambda\langle u, u\rangle = \langle \lambda u, u\rangle = \langle Fu, u\rangle \in \mathbb{R}$.
Part ii
Let $M = \sup\{|\langle Fu, u\rangle| \mid u \in H, \|u\| = 1\}$. For all $u \in H$, with $\|u\| = 1$, we have $|\langle Fu, u\rangle| \leq \|Fu\|\|u\| \leq \|F\|\|u\|\|u\| = \|F\|$, hence $M \leq \|F\|$.
To show $\|F\| \leq M$, use a formula similar to the Polarization identity. Verify that for all $u, v, \in H$ we have

$$(\langle F(u+v), u+v\rangle - \langle F(u-v), u-v\rangle) + i\left(\langle F(u+iv), u+iv\rangle - \langle F(u-iv), u-iv\rangle\right) = 4\langle Fu, v\rangle$$

Part i says all inner products of the above are real, (this is an error that prof made, but im just going to ignore it) so

$$\langle Fu, v\rangle = \frac{1}{4}\left(\langle F(u+v), u+v\rangle - \langle F(u-v), u-v\rangle\right)$$

since $|\langle Fu, u\rangle| \leq M, \forall u \in H, \|u\| = 1$, So it follows that $|\langle Fw, w\rangle| \leq M\|w\|^2, \forall w \in H$. Applying this with $w = u \pm v$, using parallelogram law, get

$$|\langle Fu, v\rangle| \leq \frac{1}{4}\left(|\langle F(u+v), u+v\rangle| + |\langle F(u-v), u-v\rangle|\right)$$

$$\leq \frac{M}{4}(\|u+v\|^2 + \|u-v\|^2) = \frac{M}{2}(\|u\|^2 + \|v\|^2)$$

Particularly when $\|u\| = \|v\| = 1$, $\langle Fu, v\rangle \in \mathbb{R}$, then $|\langle Fu, v\rangle| \leq M$. So we have $\forall u \in H, \|u\| = 1$, if $Fu = 0$ then $\|Fu\| \leq M$, for $Fu \neq 0$, $\|Fu\| = |\langle Fu, \frac{Fu}{\|Fu\|}\rangle| \leq M$. So $\|F\| \leq M$. This shows $\|F\| = M$.
Finally when λ is an eigenvalue of F and u is its correpsonding eigenvector with $\|u\| = 1$, we have $|\lambda| = |\lambda\langle u, u\rangle| = |(|\langle \lambda u, u\rangle) = |\langle Fu, u\rangle| \leq \|F\|$.

□

Theorem 2.19:
Let H be a Hilbert space and let $F : H \to H$ be a compact self-adjoint operator. Then F has an eigenvalue λ with $|\lambda| = \|F\|$.

Proof:
When $F = 0, \lambda = \|F\| = 0$ is an eigenvalue of F. Suppose $F \neq 0$, since F is self adjoint, we know that $\langle Fu, u\rangle \in \mathbb{R}, \forall u \in H$, with $\|F\| = \sup\{|\langle Fu, u\rangle| \mid \|u\| = 1\}$. So either $\|F\| = \lambda$ where $\lambda = \sup\{\langle Fu, u\rangle \mid \|u\| = 1\} > 0$ or $\|F\| = -\lambda$ where $\lambda = \inf\{\langle Fu, u\rangle \mid \|u\| = 1\} < 0$.
Now our goal is to find such vector such it proves that λ indeed is an eigenvalue. It is done by taking weakly convergent subsequence and taking the limit w as the eigenvector.
Suppose former, (latter case has similar proof), since $\lambda = \sup\big\{\langle Fu, u\rangle \mid \|u\|^2 = 1\big\}$, we can pick sequence u_n in H with $\|u_n\| = 1$ such that $\langle Fu_n, u_n\rangle \to \lambda$ in $\mathbb{R}$. Since u_n is bounded, we can pick a weakly convergent sequence $(v_k) = (u_{n_k})$ such that $v_k \to w$ weakly in H. Note that each $\|v_k\| = 1$, so $\langle Fv_k, v_k\rangle \in \mathbb{R}$ for all k with $\langle Fv_k, v_k\rangle \to \lambda$ in $\mathbb{R}$, and $\lambda = \|F\|$.
So

$$\begin{aligned} \|Fv_k - \lambda v_k\|^2 &= \|Fv_k\|^2 - 2\operatorname{Re}\langle Fv_k, \lambda v_k\rangle + \|\lambda v_k\|^2 \\ &= \|Fv_k\|^2 - 2\lambda\langle Fv_k, v_k\rangle + \lambda^2 \\ &\leq \|F\|^2 - 2\lambda\langle Fv_k, v_k\rangle + \lambda^2 \to \|F\|^2 - \lambda^2 = 0 \end{aligned}$$

Since $v_k \to w$ weakly in H and F is compact, $Fv_k \to Fw$ in H, so

$$\lambda v_k = (\lambda v_k - Fv_k) + Fv_k \to 0 + Fw = Fw$$

Since F is continuous, $F(Fw) = F(\lim_{k\to\infty} \lambda v_k) = \lambda \lim_{k\to\infty} Fv_k = \lambda Fw$
So λ is an eigenvalue of F with eigenvector Fw. □

Example 2 Notes omitted. Proof Omitted □

Theorem 2.20 (The Spectral Theorem for Compact Self-Adjoint Operators):
Let H be a Hilbert space and let $F : H \to H$ be a nonzero compact self-adjoint operator on H. The set of nonzero eigenvalues of H is at most countable, and the eigenspace of each nonzero eigenvalue is finite dimensional. When H has finitely many nonzero eigenvalues, say $\lambda_1, \ldots \lambda_n$, we have $D = \lambda_1 P_{\lambda_1} + \ldots + \lambda_n P_{\lambda_n}$, where P_{λ_k} is the orthogonal projection onto the eigenspace E_{λ_k}. When H has countably many nonzero eigenvalues, they can be arrange dinto a sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ in nonincreasing order of absolute value with $\lambda_n \to 0$, and in the space of bounded linear operators on H, we have

$$F = \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k}$$

where P_{λ_k} is the orthogonal projection onto the eigenspace E_{λ_k}.

Proof: Proof Omitted □

3 Section 3. Banach Spaces

3.1 Finite dimensional normed linear spaces

Example:
Recall that when U and V are nontrivial finite dimensional inner product spaces over $\mathbb{R}$, and $F : U \to V$ is a linear map, the closed unit ball in U is compact. (So that $\|Fx\|$ attains its maximum on the closed unit ball.)
We have

$$\|F\| = \max\{\|Fx\| \mid x \in U, \|x\| = 1\} = \|Fu\| = \sqrt{\lambda}$$

Lambda is the largest eigenvalue $F^*F : U \to U$ and u is a unit eigenvector for λ.

Theorem (3.2):
Let U be an n- dimensional normed linear space over $\mathbb{R}$. Let $\{u_1, \dots, u_n\}$ be any basis for U and let $F : \mathbb{R}^n \to U$ be the associated vector space isomorphism given by $F(t) = \sum_{k=1}^n t_k u_k$. Then, both F and F^{-1} are Lipschitz continuous.

Proof:
Showing F is continuous Let $M = \left(\sum_{k=1}^n \|u_k\|^2\right)^{\frac{1}{2}}$. For $t \in \mathbb{R}^n$, we have

$$\begin{aligned} \|F(t)\| = \left\|\sum_{k=1}^n t_k u_k\right\| &\le \sum_{k=1}^n |t_k|\|u_k\|, \text{ triangle inequality} \\ &\le \left(\sum_{k=1}^n t_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n \|u_k\|\right)^{\frac{1}{2}} \quad \text{cauchy schwarz inequality} \\ &= M\|t\| \end{aligned}$$

□

Hence, for all $s, t \in \mathbb{R}^n$, $\|F(s) - F(t)\| = \|F(s-t)\| \le M\|s-t\|$, so F is Lipschitz continuous.

Showing F^{-1} is continuous
Note that the map $N : U \to \mathbb{R}$ given by $N(x) = \|x\|$ is uniformly continuous, indeed we can take $\delta = \epsilon$ in the definition of continuity. Since F and N are both continuous, so is their composite function $G = N \circ F : \mathbb{R}^n \to \mathbb{R}$, which is given by $G(t) = \|F(t)\|$. By the extreme value theorem, the map G attains its minimum value on the unit sphere $\{t \in \mathbb{R}^n \mid \|t\| = 1\}$, which is compact.
Let $m = \min_{\|t\|=1} G(t) = \min_{\|t\|=1} \|F(t)\|$. (Operator norm of G), note that $m > 0$ because when $t \neq 0$, $F(t) \neq 0$ (since F is bijective and linear). Hence $\|F(t)\| \neq 0$. For $t \in \mathbb{R}^n$, if $\|t\| > 1$, then we have $\left\|\frac{t}{\|t\|}\right\| = 1$, so by the choice of m,

$$\|F(t)\| = \|t\| \left\|F\left(\frac{t}{\|t\|}\right)\right\| \ge \|t\| \cdot m > m$$

It follows that for all $t \in \mathbb{R}^n$, if $\|F(t)\| \le m$ then $\|t\| \le 1$. Since F is bijective, it follows that for $x \in U$, if $\|x\| \le m$, then $\|F^{-1}(x)\| \le 1$. So for all $x \in U$, if $x = 0$ then $\|F^{-1}(x)\| = 0 = \|x\|/m$. If $x \neq 0$ then since

$\left\| \frac{mx}{\|x\|} \right\| = m$, we have

$$\|F^{-1}(x)\| = \frac{\|x\|}{m} \left\| F^{-1}\left(\frac{mx}{\|x\|}\right) \right\| \leq \frac{\|x\|}{m}$$

So, for all $x, y \in U$ we have $\|F^{-1}(x) - F^{-1}(y)\| = \|F^{-1}(x-y)\| \leq \frac{1}{m}\|x-y\|$. So F^{-1} is Lipschitz continuous.

Corollary (3.3):
When U and V are finite-dimensional normed linear spaces, every linear map $F : U \to V$ is Lipschitz continuous.

Corollary (3.4):
When U is a finite-dimensional vector space, any two norms on U induce the same topology, and a sequence converges in one norm if and only if it converges in the other.

Definition (3.5): Let Y be a metric space and $\varnothing \neq X \subseteq Y$. Recall that for $y \in Y$, we defined distance between y and X to be

$$d(y, X) = \inf\{d(y, x) \mid x \in X\}$$

Recall or prove that when X is compact, the minimum value of $d(y, x), x \in X$ is attained and so we can pick $x \in X$ such that $d(y, x) = d(y, X)$.

Theorem (3.6):
Let W be a normed linear space and let $U \subseteq W$ be a finite dimensional subspace. Then for every $w \in W$, there exists $u \in U$ such that $d(w, u) = d(w, U)$.

Proof: Let $w \in W$. If $w \in U$ we can take $u = w$ so that $d(w, u) = 0 = d(w, U)$. Suppose $w \notin U$. let $d = d(w, U)$, and note that since U is closed, we have $d > 0$. (since we can choose $r > 0$ so that $B(w, r) \cap U = \varnothing$ and then $d \geq r$.)
Let $K = \overline{B}(w, d+1) \cap U$. Note that $d(w, K) = d(w, U)$. Indeed, since $K \subseteq U$ we have $d(w, K) \geq d(w, U) = d$ and that on the other hand, for any $0 < \epsilon < 1$ we can choose $u \in U$ with $d \leq d(w, u) < d + \epsilon < d + 1$, and then we have $u \in K$ hence $d(w, K) \leq d(w, u) < d + \epsilon$. Since K is closed and bounded in U, and U is a finite dimensional vector space (so we have a bijective map $F : \mathbb{R}^n \to U$ with F and F^{-1} both Lipschitz continuous). It follows that K is compact. Since K is compact, we can choose $u \in K$ such that $d(w, u) = d(w, K) = d(w, U)$.

□

Lemma 3.1 (3.7):
Let W be a normed linear space and let $U \subsetneq W$ be a proper closed subspace. For every $0 < r < 1$, there exists an element $w \in W \setminus U$ with $\|w\| = 1$ and that $d(w, U) \geq r$.

Proof: Let $0 < r < 1$. Since $U \subsetneq W$ we can choose $v \in W \setminus U$. Let $d = d(v, U)$ and note that since U is closed, we have $d > 0$. Since $d = \inf\{\|u - v\| \mid u \in U\}$, we can choose $u \in U$ such that $d \leq \|v - u\| < \frac{d}{r}$. Let $w = \frac{v-u}{\|v-u\|}$. Then we have $\|w\| = 1$ and for all $x \in U$ we have

$$\|x - w\| = \left\| x - \frac{v-u}{\|v-u\|} \right\| = \frac{1}{\|v-u\|} \cdot \| \|v-u\|x + u - v\| \geq \frac{r}{d} \cdot d = r$$

□

> **Theorem (3.8. Riesz's Theorem):**
> Let U be a normed linear space. Then U is finite dimensional if and only if the closed unit ball in U is compact.

Proof: Suppose that U is finite dimensional. Let $\mathcal{B} = \{u_1, \dots u_n\}$ be a basis for U and let $F : \mathbb{R}^n \to U$ be the isomorphism given by $F(t) = \sum_{k=1}^{n} t_k u_k$. By Theorem 3.2, F and F^{-1} are continuous. Since F is continuous, $F^{-1}(\overline{B}(0,1))$ is closed, and since F^{-1} is continuous, $F^{-1}(\overline{B}(0,1))$ is bounded by theorem 1.26. (i.e. Operator is bounded if and only if Lipschitz continuous). Since $F^{-1}(\overline{B}(0,1))$ is closed and bounded in $\mathbb{R}^n$, it is compact. Since F is a homeomorphism (homeomorphism is continuous function with a continuous inverse) and $F^{-1}(\overline{B}(0,1))$ is compact, it follows that $(\overline{B}(0,1))$ is also compact.

Suppose that U is infinite dimensional. Choose $u_1 \in U$ with $\|u_1\| = 1$ and let $U_1 = \text{Span}\{u_1\}$. Since U_1 is finite dimensional, it is closed and it is a proper subspace of U so by the above lemma we can pick $u_2 \in U \setminus U_1$ with $\|u_2\| = 1$ with $d(u_2, U_1) \geq \frac{1}{2}$. Note that this implies that $d(u_2, u_1) \geq \frac{1}{2}$. Let $U_2 = \text{Span}\{u_1, u_2\}$, and note that U_2 is a proper closed subspace of U. By lemma again, pick $u_3 \in U \setminus U_2$ with $\|u_3\| = 1$ such that $d(u_3, U_2) \geq \frac{1}{2}$. Note that this implies that $d(u_3, u_1) \geq \frac{1}{2}$ and that $d(u_3, u_2) \geq \frac{1}{2}$. Repeat this procedure to obtain a sequence $(u_n)_{n \geq 1}$ such that $\|u_n\| = 1$ for all $n \in \mathbb{Z}^+$ and $d(u_n, u_k) \geq \frac{1}{2}$ for $1 \leq k < n$. Then u_n is a sequence in the closed unit ball $\overline{B}(0,1)$, which has no convergent subsequences, so $\overline{B}(0,1)$ is not compact.

□

The Hahn- Banach Theorem

Definition:
Let W be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $p : W \to \mathbb{R}$. We say that p is **subadditive** when $p(x, y) \leq p(x) + p(y)$, for all $x, y \in W$. We say that p is **homogeneous** when $p(tx) = |t|p(x)$ for all $x \in W$ and all $t \in \mathbb{C}$. We say that p is **positively homogeneous** when $p(tx) = tp(x)$ for all $x \in W$ and all $t \in \mathbb{R}$, with $t \geq 0$. A **seminorm** on a vector space W is a subadditive homogeneous map $p : W \to \mathbb{R}$.

> **Theorem (3.10. The Hahn-Banach Theorem for Real Vector Spaces):**
> Let W be a vector space over $\mathbb{R}$. Let $U \subseteq W$ be a subspace, and let $p : W \to \mathbb{R}$ be subadditive and positively homogeneous. Then every linear map $f : U \to \mathbb{R}$ with $f(x) \leq p(x)$ for all $x \in U$ extends to a linear map $g : W \to \mathbb{R}$ with $g(x) \leq p(x)$ for all $x \in W$.

Idea: in $\mathbb{R}$ if you have an "almost" seminorm then you can extend any linear function from U to W that makes sure the image is "bounded" in the sense of that "almost" seminorm.

Proof: We claim that when $u \in W \setminus U$ and $V = U + \text{Span}\{w\} = \{u + tw \mid u \in U, t \in \mathbb{R}\}$, every linear map $f : U \to \mathbb{R}$ with $f(x) \le p(x)$ for all $x \in U$ extends to a linear map $g : V \to \mathbb{R}$ with $g(x) \le p(x)$ for all $x \in V$. Note that such an extension g is determined by the value $g(w) \in \mathbb{R}$ and must be given by $g(u + tw) = g(t) + tg(w)$ for all $u \in U$ and $t \in \mathbb{R}$. We shall choose $r = g(w) \in \mathbb{R}$ so that the map $g(u + tw) = f(u) + tr$ satisfies the requirement $g(v) \le p(v)$ for all $v = u + tw \in V$. Note that for all $x, y \in U$, we have

$$f(x) - f(y) = f(x - y) \le p(x - y) = p((x + w) + (-y - w)) \le p(x + w) + p(-y - w)$$

by subadditivity, and hence $-p(-y - w) - f(y) \le p(x + w) - f(x)$. It follows that

$$\sup\{-p(-y - w) - f(y) \mid y \in U\} \le \inf\{p(x + w) - f(x) \mid x \in U\}$$

so we can choose $r \in \mathbb{R}$ such that

$$-p(-y - w) - f(y) \le r \le p(x + w) - f(x) \text{ for all } x, y \in U$$

We define $g : V \to \mathbb{R}$ by $g(u+tw) = f(u)+tr$ for all $u \in U$ and $t \in \mathbb{R}$. We must show that $g(u+tw) \le p(u+tw)$ for all $u \in U$ and $t \in \mathbb{R}$. Let $u \in U$ and $t \in \mathbb{R}$.

- If $t = 0$ then we have $g(u + tw) = g(u) = f(u) \le p(u) = p(u + tw)$.
- If $t > 0$ then since $r \le p(u/t + w) - f(u/t) = \frac{1}{t}(p(u + tw) - f(u))$ by positive homogeneity, then we have $tr \le p(u + tw) - f(u)$ hence $g(u + tw) = f(u) + tr \le p(u + tw)$.
- If $t < 0$ then since $r \ge -p(-\frac{u}{t} - w) - f(u/t) = \frac{1}{t}(p(u+tw) - f(u))$ (by positive homogeneity) Is it just me or the arithmetic is weird? we have $tr \le p(u+tw) - f(u)$, hence $g(u+tw) = f(u)+tr \le p(u+tw)$, as required.

This completes the proof of our claim.
We now complete the actual proof by using Zorn's lemma. Let S be the set of all linear extensions of f dominated by p, (that is the set of all linear maps $g : V \to \mathbb{R}$ where V is a subspace of W containing U, such that $g(x) = f(x)$ for all $x \in U$ and $g(x) \le p(x)$ for all $x \in V$.) Define an order on S by stipulating that $g_1 \le g_2$ when g_2 is an extension of g_1. (equivalently the graph of g_2 contains that graph of g_1.) Note that every chain $C = \{g_\alpha : V_\alpha \to \mathbb{R} \mid \alpha \in A\}$ in S has an upper bound, namely the map $g : V \to \mathbb{R}$ where $V = \bigcup_{\alpha \in A} V_\alpha$, given by $g(x) = g_\alpha(x)$ for any $\alpha \in A$ for which $x \in V_\alpha$. (Verify as an exercise that the map g is well defined and linear with $g(x) = f(x)$ for all $x \in U$ and $g(x) \le p(x)$ for all $x \in V$, and g is an upper bound for C.) By Zorn's lemma, S has a maximal element $g : V \to \mathbb{R}$. By our previous claim, if we had $V \subsetneqq W$ we could choose $w \in W \setminus V$ and extend g to a linear map h defined on $V' = V + \text{Span}\{w\}$ with $h(x) \le p(x)$ for all $x \in V'$. But this would contradict the maximality of g in S. Thus we must have $V = W$ and so the maximal element g in S is an extension of f to all of W.

□

Theorem (3.11. The Hahn-Banach Theorem for Real or Complex Vector Spaces): Let W be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subseteq W$ be a subspace, let $p : W \to \mathbb{R}$ be a seminorm. Then every linear map $f : U \to \mathbb{R}$ with $|f(x)| \le p(x)$ for all $x \in U$ extends to a linear map $g : W \to \mathbb{F}$ with $|g(x)| \le p(x)$ for all $x \in W$.

Proof:
In the case where $\mathbb{F} = \mathbb{R}$ this follows immediately from the Hahn-Banach theorem for real vector spaces, because we can extend $f : U \to \mathbb{R}$ to a linear map $g : W \to \mathbb{R}$ with $g(x) \leq p(x)$ for all $x \in W$, and then (since g is linear and p is a seminorm,) we also have $-g(x) = g(-x) \leq p(-x) = p(x)$ so that $|g(x)| \leq p(x)$ for all $x \in W$.
Suppose that $\mathbb{F} = \mathbb{C}$. Let $f : U \to \mathbb{C}$ be $\mathbb{C}$ linear with $|f(x)| \leq p(x)$ for all $x \in W$. Write $f(x) = u(x) + iv(x)$ where $u, v : U \to \mathbb{R}$ and note that u, v are $\mathbb{R}$ linear and we have

$$u(ix) = Re(f(ix)) = Re(if(x)) = Re(i(u(x) + iv(x))) = -v(x)$$

So that $v(x) = -u(ix)$ and $f(x) = u(x) + iv(x) = u(x) - iu(ix)$. Since $u(x) \leq |g(x)| \leq p(x)$ for all $x \in U$, using the Hahn-Banach theorem for real vector spaces, we can extend u to an $\mathbb{R}$ linear map $w : W \to \mathbb{R}$ with $w(x) \leq p(x)$ for all $x \in W$. Define $g : W \to \mathbb{C}$ by $g(x) = w(x) - iw(ix)$. Verify that g is $\mathbb{C}$- linear and note that g extends f because $f(x) = u(x) - iu(ix)$ for all $x \in U$.
It remains to show that $|g(x)| \leq p(x)$ for all $x \in W$. Let $x \in W$. Write $g(x) = re^{i\theta}$ with $r > 0$ so that $|g(x)| = r = e^{-i\theta}g(x) = g(e^{-i\theta}x)$.
Then we have

$$|g(x)| = Re(|g(x)|) = Re(g(e^{-i\theta}x)) = w(e^{-i\theta}x) \leq p(e^{-i\theta}x) = p(x)$$

as required. □

Theorem (3.12. The Hahn- Banach Theorem for Bounded Linear Functionals):
Let W be a normed liner space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $U \subseteq W$ be a subspace. Then every bounded linear map $f \in U^*$ extends to a bounded linear map $g \in W^*$ with $\|g\| = \|f\|$.

Proof:
Let $f \in U^*$, that is let $f : U \to \mathbb{F}$ be a bounded linear map. Define $p : W \to \mathbb{R}$ by $p(x) = \|f\|\|x\|$. Then $p(x+y) = \|f\|\|x+y\| \leq \|f\|(\|x\| + \|y\|) = p(x) + p(y)$, and $p(tx) = \|f\|\|tx\| = |t|\|f\|\|x\| = |t|p(x)$, so p is a seminorm. By the above theorem, we can extend f to a linear map $g : W \to \mathbb{F}$ with $|g(x)| \leq p(x) = \|f\|\|x\|$ for all $x \in W$. Since $|g(x)| \leq \|f\|\|x\|$ for all $x \in W$, we have $\|g\| \leq \|f\|$ (in particular g is bounded linear). Since $g(x) = f(x)$ for all $x \in U$, we have $\|g\| = \sup\{|g(x)| \mid x \in W, \|x\| = 1\} \geq \sup\{|g(x) \mid x \in U, \|x\| = 1|\} = \|f\|$.

□

Corollary (3.13):
Let W be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $0 \neq w \in W$. Then there exists a bounded linear functional $g \in W^*$ with $g(w) = \|w\|$ and $\|g\| = 1$.

Proof: Let $U = \text{Span}\{w\}$ and define $f : U \to \mathbb{F}$ by $f(tw) = t\|w\|$. Then $f \in U^*$ with $f(w) = \|w\|$ and $\|f\| = 1$. By the Hahn-Banach theorem, (for bounded linear functionals), f extends to a bounded linear functional $g \in W^*$ with $\|g\| = \|f\| = 1$.

□

Corollary (3.14):
Let W be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $U \subsetneq W$ be a proper closed subspace, and let $w \in W \setminus U$. Then there exists a bounded linear functional $g \in W^*$ with $\|g\| = 1$ such that $g(w) = d(w, U)$ and $g(u) = 0$ for all $u \in U$.

Proof:
Let $d = d(w, U)$ and note that $d > 0$ because U is closed and $w \notin U$. Let $V = U + \text{Span}\{w\} = \{u + tw \mid u \in U, t \in \mathbb{F}\}$. Define $f \in V^*$ by $f(u + tw) = td$. Note that $f(u) = 0$ for all $u \in U$ and $f(w) = d$. We claim that $\|f\| = 1$. Recall or verify that for all $t \in \mathbb{F}$, we have $d(tw, U) = |t|d(w, U)$. It follows that for all $u \in U$ and $t \in \mathbb{F}$ we have $|f(u + tw)| = |t|d = |t|d(w, U) = d(tw, U) \leq d(tw, -u) = \|u + tw\|$ so $\|f\| \leq 1$. On the other hand, for all $0 < r < 1$, since $d = d(w, U) = \inf\{d(w, x) \mid x \in U\}$, we can choose $u \in U$ such that $d \leq d(w, -u) < \frac{d}{r}$ and then we have $|f(u + w)| = d > rd(w, -u) = r\|u + w\|$ and hence $\|f\| > r$. Thus $\|f\| = 1$ as claimed. By the Hahn-Banach theorem we can extend $f = V^*$ to a bounded linear map $g \in W^*$ with $\|g\| = \|f\| = 1$. □

Corollary (3.15):
Let W be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If W^* is separable, then W is separable.

Proof: Suppose that W^* is separable. Choose a sequence $(f_n)_{n \geq 1}$ in W^* such that the set $\{f_n \mid n \in \mathbb{Z}^+\}$ is dense in W^*. For each $n \in \mathbb{Z}^+$ choose $u_n \in W$ with $\|u_n\| = 1$ such that $|f_n(u_n)| > \frac{1}{2}\|f_n\|$. (always exist because operator norm is an upper bound for $\|f_n(u_n)\|$). Let $U = \text{Span}\{u_n \mid n \in \mathbb{Z}^+\}$. Recall (or verify) that when $U \subseteq W$ is a subspace of a normed linear space W, the closure $\overline{U}$ of U in W is also a subspace. We claim that $\overline{U} = W$. (which would show that W is separable.)
Suppose towards contradiction that $\overline{U} \subsetneq W$. Choose $w \in W \setminus \overline{U}$. By corollary 3.14, we can choose $g \in W^*$ such that $\|g\| = 1$ and $g(w) = d(w, \overline{U})$ and $g(v) = 0$ for all $v \in \overline{U}$. In particular, note that $g(u_n) = 0$ for all $n \in \mathbb{Z}^+$. Since $\{f_n \mid n \in \mathbb{Z}^+\}$ is dense in W^* we can choose an index $n \in \mathbb{Z}^+$ such that $\|f_n - g\| < \frac{1}{3}$. Then we have

$$1 = \|g\| = \|g - f_n + f_n\| \leq \|g - f_n\| + \|f_n\| < \frac{1}{3} + \|f_n\|$$

hence $\|f_n\| \geq \frac{2}{3}$. Since $\|f_n\| \geq \frac{2}{3}$, $|f_n(u_n)| > \frac{1}{2}\|f_n\|$, $g(u_n) = 0$, $\|u_n\| = 1$ and $\|f_n - g\| < \frac{1}{3}$, we have

$$\frac{1}{3} < \frac{1}{2}\|f_n\| < |f_n(u_n)| = |f_n(u_n) - g(u_n)| = |(f_n - g)(u_n)| \leq \|f_n - g\| < \frac{1}{3}$$

which gives the desired contradiction. Thus $\overline{U} = W$ as claimed.
Finally, note that when $\mathbb{F} = \mathbb{R}$, the set $\text{Span}_{\mathbb{Q}}\{u_1, u_2, \ldots\}$ is countable and dense in U hence also dense in $\overline{U} = W$, and when $\mathbb{F} = \mathbb{C}$, the set $\text{Span}_{\mathbb{Q}[i]}\{u_1, u_2, \ldots\}$ is countable and dense in U, hence also in $\overline{U} = W$. □

The Hahn- Banach Separation Theorem

Definition:
Let U be a real vector space and let $A \subset U$. A point $a \in A$ is called an **internal point** of A when for every $u \in U$, there exists $r > 0$ such that $a + tu \in A$ for all $t \in (-r, r)$. The set of internal points of A is called the **core** of A, and is denoted by Core(A). Note that when U is a normed linear space, the interior of A is contained in the core of A.

Definition: Let U be a real vector space. Let $A \subseteq U$ be convex with $0 \in \text{Core}(A)$. We defined the **Minkowski functional** of A to be the map $p = p_A : U \to \mathbb{R}$ given by

$$p(x) = \inf\{r > 0 \mid \frac{1}{r}x \in A\}$$

Note that the set $\{r > 0 \mid \frac{1}{r}x \in A\}$ is nonempty because $0 \in \text{Core}(A)$.

> **Theorem (3.19 The Minkowski Functional):**
> Let U be a real vector space and let $A \subseteq U$ be convex with $0 \in \text{Core}(A)$. Then the Minkowski functional of A is positively homogeneous and subadditive.

Proof: Let $p = p_A$ be the Minkowski functional of A. Then p is positively homogeneous because for $x \in U$, $t \geq 0$, we have

$$p(tx) = \inf\{r > 0 \mid \frac{1}{r}tx \in A\} = \inf\{ts \mid s > 0, \frac{1}{s}x \in A\} = t \cdot \inf\{s > 0 \mid \frac{1}{s}x \in A\} = tp(x)$$

To show that p is subadditive, let $x, y \in U$ and let $\epsilon > 0$. Choose $s \in S = \{r > 0 \mid \frac{1}{r}x \in A\}$ such that $p(x) \leq s < p(x) + \frac{\epsilon}{2}$ and choose $t \in T = \{r > 0 \mid \frac{1}{r}y \in A\}$ with $p(y) \leq t < p(y)\frac{\epsilon}{2}$. Since $\frac{1}{s}x \in A$ and $\frac{1}{t}y \in A$ and A is convex, we have

$$\frac{1}{s+t}(x+y) = \frac{s}{s+t} \cdot \frac{1}{s}x + \frac{t}{s+t} \cdot \frac{1}{t}y \in A$$

so that $s + t \in R = \{r > 0 \mid \frac{1}{r}(x+y) \in A\}$. Thus $p(x+y) = \inf R \leq s + t < p(x) + p(y) + \epsilon$. Since $p(x+y) < p(x) + p(y) + \epsilon$, for all $\epsilon > 0$, it follows that $p(x+y) \leq p(x) + p(y)$.

□

> **Theorem (3.20 The Hahn-Banach Separation Theorem):**
> Let U be a real vector space. Let A and B be disjoint nonempty convex subsets of U, with $\text{Core}(A) \neq \varnothing$. Then there exists a nonzero linear map $f : U \to \mathbb{R}$ such that $f(x) \leq f(y)$ for every $x \in A$ and $y \in B$.

Proof:
Let $a \in \text{Core}(A)$, let $b \in B$, and let C be the convex set $C = A - B - a + b$. Note that $-B$ is convex, so is $A - B$. Removing the points a, b does not affect the convex-ness because it only removes a hole from the set. Since $A \cap B = \varnothing$ we have $0 \notin A - B$ so $b - a \notin C$. Since $a \in \text{Core}(A)$ we have $0 \in \text{Core}(A - a)$. Since $A - a \subseteq (A - a) - (B - b) = C$, we also have $0 \in \text{Core}(C)$. Let $p : U \to \mathbb{R}$ be the Minkowski functional of C, given by $p(x) = \inf\{r > 0 \mid \frac{1}{r}x \in C\}$. Since $0 \in C$ and $b - a \notin C$ and C is convex, we have $t(b - a) \notin C$ for all $t \geq 1$ and so $p(b - a) \geq 1$. On the other hand, we have $p(x) \leq 1$ for all $x \in C$.

Let $f : \text{Span}\{b - a\} \to \mathbb{R}$ to be the linear map given by $f(t(b - a)) = tp(b - a)$. When $t > 0$, since p is positively homogeneous we have $f(t(b - a)) = tp(b - a) = p(t(b - a))$ and when $t \leq 0$ since p is nonnegative we have $f(t(b - a)) = tp(b - a) \leq 0 \leq p(t(b - a))$, and so $f(x) \leq p(x)$ for all $x \in \text{Span}\{b - a\}$. By the Hahn Banach Theorem for real vector spaces, we can extend f to a linear map $f : U \to \mathbb{R}$, with $f(x) \leq p(x)$ for all $x \in U$. For all $x \in A, y \in B$ since $x - y - a + b \in C$ we have

$$1 \geq p(x - y + a - b) \geq f(x - y - a + b) = f(x) - f(y) + f(b - a) \geq f(x) - f(y) + 1$$

so that $f(x) \leq f(y)$.

□

3.1.1 The Riesz Representation Theorem

Definition: Let $a \leq b$ and let $f : [a, b] \to \mathbb{R}$. For a partition $P = (x_0, x_1 \ldots x_n)$ of $[a, b]$ (so we have $a = x_0 < x_1 < \ldots < x_n = b$) the **variation** of f for the partition P is

$$V(f, P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

and the **variation** of f on the interval $[a, b]$ is

$$V(f, [a, b]) = \sup\{V(f, P) \mid P \text{ is a partition of } [a, b]\}$$

We say that g is of **bounded** variation on $[a, b]$ when $V(f, [a, b]) < \infty$, and we write $\mathcal{BV}[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is of bounded variation}\}$

Theorem (3.23):
Let $a \leq b$ and let $f : [a, b] \to \mathbb{R}$. Then f is of bounded variation on $[a, b]$ if and only if f is rectifiable. (Meaning the graph of f has finite length).

Proof:
Recall that length of the graph f on $[a, b]$ is defined as follows. For a partition $P = (x_0, x_1 \ldots x_n)$ of $[a, b]$, we define

$$L(f, P) = \sum_{k=1}^{n} \sqrt{(f(x_k) - f(x_{k-1}))^2 + (x_k - x_{k-1})^2}$$

and then the length of the graph f on $[a,b]$ is given by

$$L(f,[a,b]) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$

For any partition $P = (x_0, \dots x_n)$ of $[a,b]$, since

$$|f(x_k) - f(x_{k-1})| \leq \sqrt{(f(x_k) - f(x_{k-1}))^2 + (x_k - x_{k-1})^2}$$

for all indices k, it follows that $V(f,P) \leq L(f,P)$.
Since $V(f,P) \leq L(f,P)$ for all partitions P, it follows that $V(f,[a,b]) \leq L(f,[a,b])$. On the other hand, for all partitions $P = (x_0, \dots, x_n)$ of $[a,b]$ we have

$$\sqrt{(f(x_k) - f(x_{k-1}))^2 + (x_k - x_{k-1})^2} \leq |f(x_k) - f(x_{k-1})| + (x_k - x_{k-1})$$

(sum of edges of a triangle bigger than the other)
for all indices k, it follows that $L(f,P) \leq V(f,P) + \sum_{k=1}^{n}(x_k - x_{k-1}) = V(f,P) + (b-a)$. Since $L(f,P) \leq V(f,P) + (b-a)$ for all P, it follows that $L(f,[a,b]) \leq V(f,[a,b]) + (b-a)$.

□

Definition: Let $g \in \mathcal{BV}[a,b]$. For a partition $P = (x_0, x_1, \dots x_n)$ of $[a,b]$, write $\|P\| = \max\{x_k - x_{k-1} \mid 1 \leq k \leq n\}$. For $f \in \mathcal{C}[a,b] = \mathcal{C}([a,b],\mathbb{R})$, we define the **Riemann-Stieltjes integral** of f on $[a,b]$ with respect to the weight function g to be

$$\int_a^b f dg = \int_a^b f(x) dg(x) = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(t_k)(g(x_k) - g(x_{k-1}))$$

This means that $\int_a^b f\ dg$ is the unique real number such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every partition $P = (x_0, \dots x_n)$ of $[a,b]$ with $\|P\| < \delta$. For all $t_1, t_2, \dots t_n$ with each $t_k \in [x_{k-1}, x_k]$ we have

$$\left| \int_a^b f\ dg - \sum_{k=1}^{n} f(t_k)(g(x_k) - g(x_{k-1})) \right| < \epsilon$$

Theorem (3.26 The Riesz Representation Theorem):
For every $L \in (C[a,b])^*$, there exists $g \in \mathcal{BV}[a,b]$ with $g(a) = 0$ and $V(g,[a,b]) = \|L\|$ such that for all $f \in \mathcal{C}[a,b]$ we have

$$L(f) = \int_a^b f\ dg$$

Proof: Proof omitted

□

The Open Mapping Theorem and the Closed Graph Theorem

Theorem (3.27 The Open Mapping Theorem):
Let U and V be Banach spaces. Let $F \in \mathcal{B}(U, V)$ be surjective. Then F is open (meaning that the set $FA = \{Fa \mid a \in A\}$ is open in V for every open set A in U.)

Proof: Proof omitted □

Definition: Let X be a metric space equipped with two metrics d_1, d_2. We say the two metrics are **equivalent** when they induce the same topology on X, that is when every open ball $B_2(a, \epsilon)$ contains an open ball $B_1(a, \delta)$ and vice versa. Equivalently, the two metrics are equivalent when the identity map $I : (X, d_1) \to (X, d_2)$ is a homeomorphism.
Similarly, when U is a vector space which equipped with two norms $\|\|_1, \|\|_2$, we say the two norms are **equivalent** if they induce the same topology on U. By theorem 1.26, the norms are equivalent when there exists $\ell, m \geq 0$ such that for all $x \in U$ we have $\|x\|_2 \leq \ell \|x\|_1$ and $\|x\|_1 \leq m\|x\|_2$.

Corollary (3.29):
Let U be a vector space equipped with two norms $\|\|_1, \|\|_2$. Suppose that U is complete under both norms. If there exists $\ell \geq 0$ such that $\|x\|_2 \leq \ell \|x\|_1$ for all $x \in U$, then the two norms are equivalent. Equivalently if the identity map $I : (U, d_1) \to (U, d_2)$ is continuous, then it is a homeomorphism.

Definition: When X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology given by taking the basic open sets to be the sets of the form $A \times B$ when A open in X and B open in Y. When X, Y are metric spaces, there are various way one can define a metric on $X \times Y$ so that the induced topology is the product topology. Let us define the **product metric** on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$$

Not only does this metric induces product topology on $X \times Y$, it also behaves as expected with sequences. That is if (x_n) is a sequence in X and (y_n) is a sequence in Y and $a \in X$ and $b \in Y$, then $(x_n, y_n) \to (a, b)$ in $X \times Y$ if and only if $x_n \to a$ in X and $y_n \to b$ in Y. Also, (x_n, y_n) is cauchy in $X \times Y$ if and only if (x_n) is cauchy in X and (y_n) is Cauchy in Y.

Definition: Let X and Y be metric spaces and let $f : A \subseteq X \to Y$ be a function. The **graph** of f is the set

$$\text{Graph}(f) = \{(x, y) \mid x \in A, y = f(x)\} = \{(a, f(a)) \mid a \in A\}$$

We say that f has a **closed graph** (or simply that f is closed) when the graph of f is closed in $X \times Y$ using the product topology. Note that F has a closed graph when for every sequence (x_n) in A,

$$\text{if } x_n \to a \in X \text{ and } f(x_n) \to b \text{ in } Y \text{ then } a \in A \text{ and } b = f(a)$$

Theorem (3.32 The Closed Graph Theorem):
Let U and V be Banach spaces, and let $F : U \to V$ be a linear operator. If the graph of F is closed, then F is continuous.

Proof: proof omitted

□

Section 4. Topology

Topological Spaces and Bases

Definition (Topology): A topology is a set X on a set $\mathcal{T}$ of subsets of X such that

- $\varnothing \in \mathcal{T}$ and $X \in T$
- For any indexed set $T_\alpha, \alpha \in A$ that $\mathcal{T}_\alpha \subseteq \mathcal{T}$, $\bigcup_{\alpha \in A} \mathcal{T}_\alpha \in X$.
- For any index $1, \ldots n$, and sets $\mathcal{T}_i \subseteq \mathcal{T}$, $\bigcap_{i=1}^{n} \mathcal{T}_i \in X$.

A set X with a topology $\mathcal{T}$ is called a topological space. When X is a topological space with topology $\mathcal{T}$, a set A is open (in X with respect to $\mathcal{T}$) when $A \in \mathcal{T}$. We say A is closed (in X with respect to $\mathcal{T}$) when $A^c = X \setminus A \in \mathcal{T}$.

When X is a topological space and $A \subseteq X$, the interior of A (in X), denoted by A^0 or $Int_X(A)$, is the largest open set contained in A (union of the sets of all open sets in A which are contained in A.) The closure of A (in X), denoted by $\overline{A}$ or $Cl_X(A)$, is the smallest closed set which contains A. (Intersection of the sets of all closed sets in X that contains A.)
When S and T are two topologies of X, we say S is coarser than T or T is finer than S if $S \subseteq T$. We say S is strictly coarser than T or T is strictly finer than S when $S \subsetneq T$.
Note that intersection of nonempty set of topologies on X is also a topology on X. When X is a set and S is any set of subset of X, there is a unique coarsest topology $\mathcal{T}$ on X with $S \subseteq \mathcal{T}$ (intersection of all topologies on X that contains S), we call the topology on X generated by S.

Definition (Basis): A basis for a topology on a set X is a set $\mathcal{B}$ of subsets of X such that

1. $X = \bigcup \mathcal{B}$
2. for all $U, V \in \mathcal{B}$, and $a \in U \cap V$, there exists $W \in \mathcal{B}$ such that $a \in W \subseteq U \cap V$.

When $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$, (So the elements of $\mathcal{B}$ are open in X with respect to $\mathcal{T}$), we say that $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, and elements in $\mathcal{B}$ are the basic open sets in X.

Theorem (4.2):
Let X be a set, let $\mathcal{B}$ be a basis for a topology on X. Let T be the topology generated by $\mathcal{B}$ on X. Then for all $A \subseteq X$, we have

1. $A \in \mathcal{T}$ if and only if for every $a \in A$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq S$
2. $A \in \mathcal{T}$ if and only if A is a union of elements of B.

Proof: Let $S = \{A \subseteq X \mid \forall a \in A, \exists U \in \mathcal{B}, a \in U \subseteq A\}$.
We claim that S is a topology on X.
$\varnothing \in S$ is vacuously true and $x \in S$ because $X = \bigcup \mathcal{B}$, so given any $a \in X$ we can always choose $U \in \mathcal{B}$ with $a \in U$.
When R is any subset of S, we wan to show $\bigcup R \in S$. Indeed, given any $a \in \bigcup R$, we can choose $U \in R$ with $a \in U$ and then we have $A \in U \in R$, showing that $\bigcup R \in S$.
It remains to show that $\cap R \in S$ for every finite set $R \subseteq S$.
By induction it suffices to show that for all $A, B \in S$, $A \cap B \in S$. Let $A, B \in S$. Let $a \in A \cap B$. Since $a \in A, A \in S$, we can choose $U \in \mathcal{B}$ with $a \in U \subseteq A$. Since $a \in B, B \in S$, we can choose $V \in \mathcal{B}$ such that $a \in V \subseteq B$.
Since B is a basis, we can choose $W \in B$ such that $a \in W \subseteq U \cap V$. Then we have $a \in W \subseteq U \cap V \subseteq A \cap B$. Hence $A \cap B \in S$. Therefore S is a topology on X as claimed.
If $A \subseteq X$ then $A \in S$ if and only if A is a union of elements in $\mathcal{B}$
Suppose $A \in S$, then each $a \in A$, we can choose $U_a \in \mathcal{B}$ with $a \in U_a \subseteq A$, and we have $A = \bigcup_{a \in A} U_a$, which is a union of elements of $\mathcal{B}$. On the other hand, if A is a union of elements in $\mathcal{B}$, say $A = \bigcup R$, with $R \subseteq \mathcal{B}$, then given $a \in A$, we can choose $U \in R$ such that $a \in U$, then we have $a \in U \subseteq A$, this shows $A \in S$.
Claim that $S = \mathcal{T}$.
$\mathcal{T} \subseteq S$
Note that if $U \in \mathcal{B}$ we have $U \in S$. Since S is a topology that contains $\mathcal{B}$, $\mathcal{T}$ is therefore the coarest topology that contains $\mathcal{B}$, so $\mathcal{T} \subseteq S$.
$S \subseteq \mathcal{T}$
Every topology that contains $\mathcal{B}$ also contains all possible unions of elements in $\mathcal{B}$. It follows that $\mathcal{T}$ contains all such unions, so $S \subseteq \mathcal{T}$. So $S = \mathcal{T}$.

□

Example 3 In a metric space X, the set $\mathcal{B} = \{B_r(a) \mid a \in X, r > 0\}$ is a basis for the metric topology on X. ▫

Theorem (4.4):
Let X be a topological space with basis $\mathcal{B}$, and let $A \subseteq X$. Then for $a \in X$, we have $a \in \overline{A}$ if and only if $A \cap U \neq \varnothing$ for every $U \in \mathcal{B}$ with $a \in U$.

Example 4 Subspace topology
When Y is a topological space with topology $\mathcal{T}$, and $X \subseteq Y$, the subspace topology on X is the topology $S = \{V \cap X \mid V \in \mathcal{T}\}$.
Verify that a subset $A \subseteq X$ is closed in X if and only if there exists a closed set B in Y such that $A = B \cap X$.
Verify that a if $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on Y then $\mathcal{B} = \{V \cap X \mid V \in \mathcal{B}\}$ is a basis for the subspace topology S on X.
Recall that if Y is a metric space, $\mathcal{T}$ is the metric topology on Y, the subspace topology on X is equal to

the topology on X induced by the metric on X obtained by restricting the metric on Y.

Example 5 Product topology When X, Y are topological spaces with topologies S and $\mathcal{T}$, the product topology on $X \times Y$ is the topology with the basis $\mathcal{E} = \{U \times V \mid U \in S, V \in T\}$. We shall verify that it indeed is a topology. If $\mathcal{B}, \mathcal{C}$ are bases for the topologies on X, Y, the set $\mathcal{D} = \{U \times V \mid U \in \mathcal{B}, V \in \mathcal{C}\}$, is another basis for the topology.
Also verify that $A \subseteq X, B \subseteq Y$, the subspace topology on $A \times B$, as a subspace of $X \times Y$, using the product topology, is equal to the product topology on $A \times B$, where A, B use the subspace topologies as subspaces of X, Y. □

Example 6 Quotient topology When X is a set, $\sim$ is an equivalence relation on X, the quotient of X by $\sim$ is the set of equivalence classes

$$X/\sim = \{[a] \mid a \in X\}, \text{ where } [a] = \{x \in X \mid x \sim a\}$$

The quotient map $q : X \to X/\sim$ is the map given by $q(a) = [a]$. When X is a topological space with topology $\mathcal{T}$, the quotient topology on $X/\sim$ is the topology

$$S = \{V \subseteq X/\sim \mid q^{-1}(V) \in \mathcal{T}\} = \{V \subseteq X/\sim \mid \bigcup V \in \mathcal{T}\}$$ □

Continuous Functions and Compact Sets

Definition (Hausdorff): A topological space X is called Hausdorff when it has the property that for all $a, b \in X$ with $a \neq b$, there exists disjoint open sets $U, V \subseteq X$ with $a \in U, b \in V$. When X is Hausdorff and $a \in X$, the set $\{a\}$ is closed.
Note that all metric spaces are Hausdorff because given $a \neq b$, if we let $r = d(a, b)$, take $U = B(a, \frac{r}{2})$, $V = B(b, \frac{r}{2})$.

Definition (Continuous): Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$. Equivalently, f is continuous if and only if $f^{-1}(B)$ is closed in X for every closed set $B \subseteq Y$.

Definition (Covers): Let X, Y be topological spaces, $A \subseteq X$. An open cover for A is a set S of open sets in X such that $A \subseteq \bigcup S$. When S is an open cover for A in X, a subcover for A is a subset $T \subseteq S$ with $A \subseteq \bigcup T$. A is compact if every open cover has a finite subcover.

Theorem (4.12):
Let $A \subseteq X \subseteq Y$ where Y is a topological space. Then A is compact in X (with the subset topology inherited from Y) if and only if A is compact in Y.

Definition: Let X be a topological space. We say X has finite intersection property on closed sets when every set T of closed sets in X, if every finite subset of T has nonempty intersection, then T has non-empty intersection.

Theorem (4.15):
Let X be a metric space. Then X is compact if and only if X has the finite intersection property on closed sets.

Theorem (4.16):
Every closed subspace of a compact set is compact.

Theorem (4.17):
Every compact subspace of a Hausdorff space is closed.

Theorem (4.18):
The image of a compact space under a continuous map is compact.
Note that continuous maps do not necessarily send closed sets to closed sets.

Theorem (4.20. The Extreme Value Theorem):
A continuous map $f : X \to \mathbb{R}$ defined on a compact space X attains its maximum and values.

Theorem (4.21.):
Let X and Y be the topological spaces with X compact and Y Hausdorff. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.

Urysohn's Lemma and The Tietze Extension Theorem

Definition (Normal): A topological space X is called normal when all one-point sets are closed in X, and for all disjoint closed $A, B \subseteq X$, there exists disjoint open sets $U, V \subseteq X$ with $A \subseteq U, B \subseteq V$. We should verify that all metric spaces are normal.

Theorem (4.25. Urysohn's Lemma):
Let X be a normal topological space. For any disjoint closed sets $A, B \subseteq X$, there exists a continuous map $f : X \to [0, 1]$ with $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$.

See James Munkres for a better proof

Theorem (4.26. The Tietze Extension Theorem):
Let X be a normal topological space, let $A \subseteq X$ be closed, and let $a, b \in \mathbb{R}$ with $a < b$.

- Every continuous map $f : A \to [a, b]$ can be extended to a continuous map $g : X \to [a, b]$.
- Every continuous map $f : A \to (a, b)$ can be extended to a continuous map $g : X \to (a, b)$.

Infinite Products and Tychanoff's Theorem

Definition (4.27): Let $(X_k)_{k\in K}$ be an indexed set of sets. The Cartesian product of this indexed set is the set

$$\begin{aligned}\prod_{k\in K} x_k &= \left\{a : K \to \bigcup_{k\in K} X_k \mid a(k) \in X_k \text{ for all } k \in K\right\}\\ &= \{(a_k)_{k\in K} \mid a_k \in X_k \text{ for all } k \in K\}\end{aligned}$$

For each $\ell \in K$ we have the projection map $p_\ell : \prod_{k\in K} X_k \to X_\ell$ given by $p_\ell((a_k)_{k\in K}) = a_\ell$. (Basically just extracting the k-corresponding coordinate).
When $K = \{1, \ldots, n\}$, we write

$$(a_k)_{k\in K} = (a_1, a_2, \ldots a_n) \text{ and } \prod_{k\in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times X_2 \times \ldots \times X_n$$

When $K = \mathbb{Z}^+$ we write

$$(a_k)_{k\in K} = (a_k)_{k\geq 1} = (a_1, a_2, a_3, \ldots) \text{ and } \prod_{k\in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times X_3 \times \ldots$$

When each X_k is a topological space with topology $\mathcal{T}_k$, the box topology on the cartesian product is the topology with basis

$$\mathcal{B} = \left\{\prod_{k\in K} U_k \mid U_k \in \mathcal{T}_k\right\}$$

the **product topology** on cartesian product is the topology with basis

$$\mathcal{P} = \left\{\prod_{k\in K} U_k \mid U_k \in \mathcal{T}_k \text{ with } U_k = X_k \text{ for all but finitely many } k \in K\right\}$$

Note that when K is finite, the box and product topology are the same, but when K is infinite, the box topology is finer than the product topology.
That is, the topology takes the whole thing on infinitely many items in the index set, but only on finitely many of those members in the index set, it is finer.

Theorem (4.28):
Let X_k be a topological space and let $A \subseteq X_k$ be a subspace for each $k \in K$. Let $\prod_{k\in K} X_k$ be given the product topology.

1. If each X_k is Hausdorff so is $\prod_{k\in K} X_k$
2. On $\prod_{k\in K} A_k \subseteq \prod_{k\in K} X_k$, the product topology is equal to the subspace topology
3. We have $\overline{\prod_{k\in K} A_k} = \prod_{k\in K} \overline{A_k}$.

Analogues results hold when $\prod_{k\in K} X_k$ and $\prod_{k\in K} A_k$ are given the box topology.

Theorem (4.29):
Let X_k be a topological space for each $k \in K$, and let $\prod_{k\in K} X)k$ be given the product topology. For every topological space A and for every function $f : A \to \prod_{k\in K} X_k$, f is continuous if and only if $f_\ell : A \to X_k$ given by $f_\ell(x) = f(x)_\ell$ is continuous for all $\ell \in K$.

Theorem (4.31 Tychanoff's Theorem):
The product of any indexed set of compact spaces is compact, using the product topology.

Nets

Definition (Directed set): A **directed set** is a set K together with a binary relation $\leq$ such that

1. For all $a \in X, a \leq x$
2. For all $a, b, c \in X$, if $a \leq b, b \leq c$ then $a \leq c$ and
3. For all $a, b \in X$, there exists $c \in X$ such that $a \leq c$, and $b \leq c$.

When $a \leq b$ we also write $b \geq a$. A **net** in a topological space X is an indexed set $(x_k)_{k\in K}$ in X whose index set K is a directed set. When $(x_k)_{k\in K}$ is a net in X and $a \in X$, we say that $(x_k)_{k\geq K}$ converges to a (in X) and we write $x_k \to a$ (in X), when for every open set $U \subseteq X$ with $a \in U$, there exists $m \in K$ such that for all $k \in K$, if $k \geq m$, then $x_k \in U$.

Theorem (4.33):
In a Hausdorff topological space, the limit of a convergent net is unique.

Theorem (4.34):
Let X be a topological space, let $A \subseteq X$, and let $a \in X$. Then $a \in \overline{A}$ if and only if there is a net $(x_k)_{k\in K}$ in A with $x_k \to a$ in X.

Theorem (4.35):
Let X and Y be the topological spaces, let $A \subseteq X$, and let $f : A \subseteq X \to Y$. Then f is continuous on A (using the subspace topology in X) if and only if every $a \in A$, and every net $(x_k)_{k\in K}$ in A, if $x_k \to a$ in X, then $f(x_k) \to f(a)$ in Y.

Strong and Weak Topologies and the Banach Alaoglu Theorem

Definition: Let Y be a topological space and let $(f_k)_{k\in K}$ be an indexed set of functions $f_k : X_k \to Y$ where each X_k is a topological space. The **Final Topology** (or the Strong Topology) on Y (with respect to the indexed set $(f_k)_{k\in K}$) is the finest topology on Y such that each of the functions f_k is continuous. A subset $U \subseteq Y$ is open in the strong topology if and only if $f_k^{-1}(V)$ is open in X_k for every open set $V \subseteq X$ and every $k \in K$.

Definition: Let X be a topological space and let $(f_k)_{k\in K}$ be an indexed set of functions $f_k : X \to Y_k$ where each Y_k is a topological space. The **initial topology** (or the Weak topology) on X (with respect to the indexed sets $(f_k)_{k\in K}$) is the coarsest topology on X such that each of the functions f_k is continuous, that is the topology on X generated by the set $\{f_k^{-1}(U) \mid k \in K, U \in Y_k\}$.
Example: When Y is a topological space and $X \subseteq Y$, the subspace topology on X is equal to the initial topology on X with respect to the inclusion map.

Definition: Let U be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The **weak topology** on U is the initial topology on U with respect to U^*. (With respect to the indexed set $(f)_{f\in U^*}$), that is the topology generated by the sets of the form $f^{-1}(V)$ where $f \in U^*$ and V is an open set in $\mathbb{F}$.
The **weak-star** topology, or the **weak*** topology on U^* is the initial topology on U^* with respect to the indexed set $(F_u)_{u\in U}$ where $F_u \in U^{**}$ is given by $F_u(f) = f(u)$, that is the topology generated by the sets of the form $F_u^{-1}(V) = \{f \in U^* \mid f(u) \in V\}$, where $u \in U$ and V is an open set in $\mathbb{F}$.

Theorem (4.42):
Let U be a normed linear space.

- For every $a \in U$, and every net $(x_k)_{k\in K}$ in U, we have $x_k \to a$ in U using the weak topology if and only if $f(x_k) \to f(a)$ in $\mathbb{F}$ for every $f \in U^*$.
- For every $g \in U^*$, and every net $(f_k)_{k\in K}$ in U^*, we have $f_k \to g \in U^*$ using the weak* topology if and only if $f_k(x) \to g(x)$ in $\mathbb{F}$ for every $x \in U$.

Theorem (4.44. The Banach- Alaoglu Theorem):
For a normed linear space U, the closed unit ball $\overline{B}_{U^*}(0,1) = \{f \in U^* \mid \|f\| \leq 1\}$ is compact in U^* using the weak* topology. (Does not always hold for norm topology, but it does hold for the weak star topology.)