1 Introduction- Axiom of Choice

1.1 Set theory and AOC

Theorem 1.1 (Axiom of Choice):
Let $\mathcal{F}$ be a nonempty collection of non-empty sets, say $\mathcal{F} = \{A_\lambda : \lambda \in \Lambda\}$, where the index set Λ and each set A_λ is non-empty. Then there is a function

$$f : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$$

such that $f(\lambda) \in A_\lambda$ for each $\lambda \in \Lambda$.

This theorem is important when Λ is uncountable.

Definition 1.1 (Partial order): Let S be a set. A relation $\leq$ on S is called partial order if

- Reflexive: $x \leq x, \forall x \in S$
- Transitive: for all $x, y, z \in S$, if $x \leq y, y \leq z$ then $x \leq z$
- Anti-symmetric: for all $x, y \in S$, if $x \leq y$ and $y \leq x$ then $x = y$.

A partial order is a total order if $\forall x, y \in S, x \leq y$ or $y \leq x$.

Definition 1.2 (Well ordered): A partially ordered set $(S, \leq)$ is well ordered if every nonempty set has a smallest element.

One of the consequences of AOC is the Well ordering principle: every set can be well ordered.

Definition 1.3 (Upper bound, maximal, chain): Let $(S, \leq)$ be a partially ordered set then

- an upper bound for $T \subseteq S$ is an element $s \in S$ with $s \geq t, \forall t \in T$
- an element $s \in S$ is maximal whenever $x \in S$ and $x \geq s$ then $x = s$.
- A chain is a totally ordered subset of S.

Lemma (Zorn's lemma):
Suppose S is a non-empty, partially ordered set in which each chain has an upper bound. Then A has a maximal element.

Theorem 1.2:
TFAE:

- The well ordering principle
- AOC
- Zorn's lemma

Theorem 1.3 (Basis theorem):
Every vector space has a basis. This is proven by Zorn's lemma.

1.2 Section 2

Recall Riemann integrals. The basic idea of riemann integral is that for a bounded function $f : [a, b] \to \mathbb{R}$, if we get a partition $P : a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$, then the upper and lower riemann sums are:

$$U(f, P) = \sum_{i=1}^{n} \sup f \mid_{[x_{i-1}, x_i]} (x_i - x_{i-1})$$

$$L(f, P) = \sum_{i=1}^{n} \inf f \mid_{[x_{i-1}, x_i]} (x_i - x_{i-1})$$

U always $\geq L$. Also, with more refinements, U gets smaller and L gets bigger. Every upper sum is bigger than every lower sum.

Definition 1.4 (Riemann integrable): Given f, if

$$\inf\{U(f, P) : \text{all partitions } P\} = \sup\{L(f, P) : \text{all partitions } P\}$$

then we say f is riemann integrable over $[a, b]$. Write $R - \int_a^b f = \inf_P U(f, P)$.

Note that continuous functions, or any functions with finitely many discontinuities are riemann integrable.
Consider the characteristic function. Given $A \subseteq X$, the characteristic function of A, is $\chi_A(x)$, it's 1 if $x \in A$ it's 0 if $x \notin A$. Note that $\chi_{\mathbb{Q}}$ is not riemann integrable. They also behave poor under limits. (They're bad under function's point wise convergence. For example, given a sequence of functions, each are riemann integrable, their pointwise limit also is. But the limit of their riemann integral doesn't equal the riemann integral of the pointwise limit of function.)
Lebesgue integral extends Riemann integral. They behave better under limits. The main idea is: rather than partitioning the domain, they paritition the range.
This way, we can write function

$$f \sim \sum y_i \chi_{E_i}$$

given that E_i finely partitions the domain into sets with same y_i values.

1.3 Lebesgue outer measure

Ideally, we would like to have a measure, a function m on all subsets of $\mathbb{R}$ such that"

$$m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$$

with the following properties:

1. m(interval) is the length of the interval. $m(\varnothing) = 0$.
2. σ-additivity: with disjoint $\{E_i\}_{i=1}^n$ disjoint, then
$$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$$
3. $m(E) = m(E + y)$ for all $E \subseteq \mathbb{R}$, $y \in R$. This is called translation invariance, which is a "sliding" thing.

Monotonicity: give $A \subseteq B$ then measure of B is bigger than A. This is achieved through $\sigma-$additivity.
However, it's impossible to find a function with all these properties. This is a consequence of the Axiom of choice. Therefore, we use Lebesgue outer measure instead.
We use Lebesgue outer measure instead. Note that all the three properties cannot be satisfied, so we give up σ additivity on the powerset of $\mathbb{R}$, but instead only define σ additivity on a subset of the powerset of $\mathbb{R}$, namely the measurable sets. The σ additivity on the measurable sets is called the σ-sub-additivity.

Definition 1.5 (Lebesgue outer measure):
Given $A \subseteq \mathbb{R}$, let $\mathcal{C}(A) = \{\{I_n\}_{n=1}^{\infty}$where I_n are open covers of A by countable intervals$\}$.
$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : \{I_n\} \in \mathcal{C}\right\}$$

Proposition 1.4:
If I is an interval then $m^*(I) = \ell(I)$.

Proposition 1.5 (sigma subadditivity):
For all sets $A_k \subseteq \mathbb{R}$, we have
$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k)$$

Proposition 1.6:
If $A = \{x_n\}_{n=1}^{\infty}$ is any countable set then $m*(A) = 0$.

Proposition 1.7:
For any $A \subseteq \mathbb{R}$ and for any $\epsilon > 0$ there is an open set $O \supseteq A$ such that $m*(O) \leq m^*(A) + \epsilon$.

Note: when solving questions, a lot of the time you would find some epsilon and find the intervals that catch A within the ϵ. This starts the proof of many problems in this lesson.

1.4 Construction of Lebesgue Measure

Definition 1.6 (Lebesgue Measurable- Caratheodory definition): We say $A \subseteq \mathbb{R}$ is lebesgue measurable if for every $E \subseteq \mathbb{R}$ we have

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^C)$$

What sets are lebesgue measurable?

- $A = \mathbb{R}$ or $\varnothing$
- A^C if A is L.M.
- Any set A with $m^*(A) = 0$.
- Interval (a, ∞)

L.M. sets are an example of a σ-algebra.

Definition 1.7 (sigma algebra): A family Ω of subsets of $\mathbb{R}$ is called a $\sigma-$algebra if

- $\varnothing \in \Omega$
- If $A_1, A_2, \ldots \in \Omega$ then $\bigcup A_i \in \Omega$. Closed under countable unions.
- If $A \in \Omega$ then $A^C \in \Omega$. Closed under complements.

Note that the intersection of sigma algebras is still a sigma algebra.

Definition 1.8 (Borel sigma algebra):

- The Borel σ algebra is the intersection of all σ algebras containing all open sets in $\mathbb{R}$. It is the smallest σ algebra containing all open sets.
- A Borel set is any set in the Borel σ algebra.

Theorem 1.8 (Lebesgue masurable sets):
The set of Lebesgue measurable sets, denoted $\mathcal{M}$, is a σ algebra which contains the Borel sets and includes all sets of outer Lebesgue measure zero.
Moreover, if $A_n \in \mathbf{M}$ are disjoint then

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n)$$

That is, m^* restricted to lebesgue measurable sets is σ additive.

You can construct a lebesgue measurable set that is NOT borel and it requires the Cantor function

Remark 1 Important properties of Lebesgue measure

1. $m(\text{interval})$ = length of the interval
2. If $A \in \mathcal{M}$ then $A + t \in \mathcal{M}$ for all $t \in \mathbb{R}$ and $m(A) = m(A + t)$.
3. m is $\sigma-$additive. i.e. $A_1, A_2, \ldots \in \mathcal{M}$ and are disjoint then $m(\bigcup A_n) = \sum m(A_n)$
4. m is $\sigma-$sub-additive. i.e. $A_1, A_2, \ldots \in \mathcal{M}$ and not necessarily disjoint then $m(\bigcup A_n) \leq \sum m(A_n)$
5. If $A, B \in \mathcal{M}, A \subseteq B$ then $m(B) = m(A) + m(B \setminus A)$
6. m is monotonic
7. The set E has Lebesgue measure zero if and only if for every $\epsilon > 0$, there are open interval covers I_n with sum of lengths $< \epsilon$.
8. Measure of a compact set is finite.
9. For any $E \in \mathcal{M}$ we have
$$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$$

□

Proposition 1.9:

1. Continuity of measure: if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subseteq \bigcup_{i=1}^{\infty} A_i = A$, and each $A_i \in \mathcal{M}$, then $A \in \mathcal{M}$ and $\lim_n m(A_n) = m(A)$
2. Downward continuity of measure:if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \supseteq \bigcap_{i=1}^{\infty} A_i = A$, and each $A_i \in \mathcal{M}$, then $A \in \mathcal{M}$. In addition, if $m(A_1) < \infty$, then $\lim_n m(A_n) = m(A)$.

Why we want $m(A_1) < \infty$? because cases like $A_n = (n, \infty)$.
This shows that measurability works in limit of sets, where limit is taken by taking containment.

Theorem 1.10 (The Lebesgue's characterization of Riemann integrability):
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable over $[a, b]$ if and only if $m\{\text{discontinuities of } f\} = 0$.
Only functions that are continuous, or has finitely many discontinuities, or only countably many discontinuities, are Riemann integrable.

1.5 Measurable Functions

Definition 1.9 (Lebesgue Measurable): A function $f : X \to [-\infty, \infty]$ is called lebesgue measurable if for all $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ are lebesgue measurable. A compled valued function $f : X \to \mathbb{C}$ is Lebesgue measurable if both its real and complex parts are Lebesgue measurable.

Remark 2

1. Any constant f is measurable.
2. Any continuous, real-valued function f is measurable.
3. the above definition could also be characterized with $f^{-1}((a, \infty])$
4. $f^{-1}(\{\infty\})$ is a measurable set.

□

Proposition 1.11:
$f = \chi_E$ is measurable if and only if $E \subseteq \mathbb{R}$ is a measurable set.

Definition 1.10 (Simple functions): A simple function is of the form $f = \sum_{i=1}^{N} a_i \chi_{E_i}$ where each of the $E_i \subseteq \mathbb{R}$ are measurable and $a_i \in \mathbb{R}$.

Note how in Riemann integrals, step functions are the building blocks. In Lebesgue integrals, simple functions are like the building blocks. Note that these are measurable functions that takes on only finitely many real values in its range.

Proposition 1.12 (measurable function properties):
If f, g are real-valued, measurable functions (with the same domain), then so are $f \pm g, fg$ and f/g if $g \neq 0$.

Proposition 1.13 (Measurability behaves well under limits):
Suppose $\{f_n\}_{n=1}^{\infty}$ are measurable functions. Then so are $\sup_n f_n$ and $\inf_n f_n$. If $f_n \to f$ pointwise and $g_n \to g$ pointwise then f is measurable. Note that riemann integrable functions do not have this property (the step $\mathbb{Q}$ one).

Note that $f_n \to f$ pointwise implies

$$f = \liminf f_n = \sup_n (\inf_{k \geq n} f_k)$$

so it is measurable.

Theorem 1.14 (Simple functions are basic building blocks):
The positive, real-valued function $f : \mathbb{R} \to [0, \infty)$ is measurable if and only if there are simple functions ϕ_n, $n \in \mathbb{N}$ with $\phi_n \leq \phi_{n+1}$ and $\phi_n \to f$ pointwise. So this means we can make any measurable functions f based off of a pointwise increasing sequence of simple functions.

Proof: The proof idea is: each time do these:

1. expand image from $[0, k)$ to $[0, 2k)$
2. make the grains $1/2$ finer.

$\square$

Definition 1.11 (Almost everywhere): We say that $f = g$ almost everywhere, and write $f = g$ *a.e.* if

$$m(\{x : f(x) \neq g(x)\}) = 0$$

1.6 Lebesgue Integral

Definition 1.12 (Standard representation of a simple function): Every simple function has a unique expression written as

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where each E_i is disjoint and $a_i \neq 0$, and $\bigcup_{k=1}^{N} E_i = \mathbb{R}$.
Take convention that $0 \cdot \infty = 0$.

Definition 1.13 (Lebesgue integral over simple function): Assume $\phi \geq 0$, is a simple function and in standard representation, with $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$. Then the Lebesgue integral over of ϕ over the measurable set E is defined as:

$$\int_E \phi = \sum_{k=1}^{N} a_k m(E \cap E_k)$$

Recall how the Riemann integral is:

$$\int_a^b \phi = \sum_{k=1}^{N} a_k \ell(I_k \cap [a, b])$$

where I_k is the intervals.

Definition 1.14 (Lebesgue integral over measurable function): Suppose $f : E \subseteq \mathbb{R} \to [0, \infty]$ is measurable and E is measurable set. Then we define the lebesgue integral of f over E as

$$\int_E f = \sup\left\{ \int_E \phi : 0 \leq \phi \leq f, \phi \text{ is simple} \right\}$$

Definition 1.15 (Expand Lebesgue integral's domain to $\mathbb{R}$): For $f : E \to \mathbb{R}$, let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then, both $f^+, f^- : E \to [0, \infty)$, and $f = f^+ - f^-$. Both are measurable. Then

$$\int_E f = \int_E f^+ - \int_E f^-$$

$$\int_E |f| = \int_E f^+ + \int_E f^-$$

provided that we don't get the $\infty - \infty$ situation.

Definition 1.16 (Integrable): A measurable function f is integrable on E if $\int_E |f| < \infty$.

Definition 1.17: If $f : E \subseteq \mathbb{R} \to \mathbb{C}$ then we have the definition of f being integral and the integral value of f.

Remark 3 Properties of Lebesgue Integral

1. Monotonicity: if $0 \leq f \leq g$ then $\int_E f \leq \int_E g$.
2. $\int_E f = \int_{\mathbb{R}} f\chi_E$
3. If $m(E) = 0$ then $\int_E f = 0$ even if $f = \infty$ on E
4. For all scalars α, $\int \alpha f = \alpha \int f$
5. Triangle inequality: $|\int f| \leq \int |f|$
6. For all ϕ, ψ simple, $\int \phi + \int \psi = \int(\phi + \psi)$
7. Translational invariance of integral: $\int_{\mathbb{R}} f(x+y)dx = \int_{\mathbb{R}} f(x)dx$ for all $y \in \mathbb{R}$. □

1.7 Monotone and Dominated Convergence Theorems

Theorem 1.15 (Monotone convergence theorem):
Suppose $f_n \geq 0$ and measurable. If $f_n(x) \leq f_{n+1}(x)$ for all x, n, and if $f_n \to f$ pointwise, then

$$\lim_n \int_E f_n = \int_E \lim_n f_n = \int_E f$$

for any measurable set E.

Remark 4 The proof of the MCT needs a lemma, and the lemma shows that the main idea behind the theorem is continuity of measure. □

Proposition 1.16:
If $f, g \geq 0$ are measurable, then $\int_E f + g = \int_E g + \int_E h$

Lemma 1.17 (Fatou's Lemma):
Fatou's lemma is a consequence of MCT.
For $f_n \geq 0$ and measurable, then we have

$$\int \liminf_n f_n = \liminf_n \int f_n$$

Theorem 1.18 (Dominated Convergence Theorem):
Fatou's lemma helps to prove the Dominated Convergence Theorem.
Suppose f_n are measurable functions with $f_n \to f$ pointwise. Assume there is an integrable function g, such that $|f_n(x)| \leq g(x)$, for all x and n. Then

$$\int f = \lim_n \int f_n$$

In fact,

$$\int |f_n - f| \to 0$$

Theorem 1.19 (Relationship between Riemann and Lebesgue integrals):
If f is Riemann integrable over $[a, b]$, then f is Lebesgue integrable over $[a, b]$ and the Riemann and Lebesgue integrals coincide.

2 L_p spaces

Definition 2.1 (L_p **norm):** Take $1 \leq p < \infty$.
For now, $p \geq 1$

$$\|f\|_{L^p(E)} = \left(\int_E |f|^p\right)^{1/p}$$

Note that this is not an actual norm, because the norm would evaluate to 0 even for a non-zero function! (the functions with 0 a.e.). Therefore, we define an equivalence relation on the measurable functions on E via $f \sim g$ if $f = g$ *a.e.* on E.
Hence

$$L^p(E) = \{\text{equivalence classes of measurable functions } f : E \to \mathbb{C}, \|f\|_{L^p(E)} < \infty\}.$$

Proposition 2.1:
$L^p(E)$ is a vector space.

Definition 2.2 (∞**-norm, essential supremum):**

$$\|f\|_{L^\infty(E)} = \inf_{\text{A meas}, m(E \setminus A) = 0} \{\sup |f(x)|, x \in A\}$$

An equivalent definition is:

$$\|f\|_\infty = \inf\{\alpha \in \mathbb{R} : m(\{x : |f(x)| > \alpha\}) = 0\}$$

Note that the essential supremum of f is bounded above by the supremum of f.

Proposition 2.2:
If f is continuous on $\mathbb{R}$ then $\|f\|_{L^\infty(\mathbb{R})} = \sup_x |f(x)|$

Definition 2.3 (L^∞ **class):**

$$L^\infty(E) = \{\text{equivalent classes of measurable } f : E \to \mathbb{C}, \|f\|_\infty < \infty\}$$

2.1 Holder's inequality and Minkowski's inequality

Definition 2.4 (Conjugate pairs): The pair $p, q, 1 \leq p, q < \infty$ are conjugate pairs if $1/q + 1/p = 1$.

Theorem 2.3 (Holder's inequality):
p, q are conjugate pairs, f, g are measurable functions. Then

$$\int |fg| \leq \|f\|_p \|g\|_q = \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

In particular, if $f \in L^p, g \in L^q$ then $fg \in L^1$ and its 1 norm is bounded above by $\|f\|_p \|g\|_q$.

Remark 5 Holder's implies a special case of Cauchy Schwarts If $p = q = 2$ then

$$\left|\int f\bar{g}\right| \leq \int |f\bar{g}| \leq \|f\|_2 \|g\|_2$$

□

Consider defining an inner product by $\int f\bar{g} = \langle f, g \rangle$.

Theorem 2.4 (Using conjugate indices to determine L_p norms):
Let $1 \leq p < \infty$. Suppose q is p's conjugate index. Then for any $f \in L^p$,

$$\|f\|_p = \sup\left\{\left|\int fg\right| : \|g\|_q \leq 1\right\}$$

Doesn't this remind you of the operator norm somehow?
This is known as the triangle inequality for the L^p spaces and completes the proof that $\|\cdot\|_p$ is a norm.

Theorem 2.5 (Minkowski's inequality):
for f,g measurable, for $1 \leq p \leq \infty$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Corollary 2.6 (L^p is a metric space):
For any $1 \leq p \leq \infty$, L^p is a normed linear space and hence a metric space. Minkowski's inequality proved the triangular inequality part of this corollary.

2.2 Riesz-Fischer Theorem

Resume reading starting on Page 35

Definition 2.5 (Banach space): A Banach space is a normed linear space that is also complete. (With respect to the metric given by the norm.)

Theorem 2.7 (Riesz-Fischer Theorem):
L^p is a Banach space for all $1 \leq p \leq \infty$.

Corollary 2.8:
If $f_n \to f$ in L^p, $1 < p < \infty$, then there is a subsequence f_{n_k} that converges to f pointwise almost everywhere.

3 Lusin's Theorem and Fubini's theorem

Remark 6 Note that every bounded function is essentially bounded, so $C[a,b] \subseteq L^\infty[a,b]$. Also, by simply working out the definitions, $L^\infty[a,b] \subseteq L^p[a,b]$ for all $p < \infty$. It is natural to ask whether $C[a,b]$ is a "large" subset of L^p? Lusin's theorem addresses this question. $\square$

Theorem 3.1 (Lusin's theorem):
The continuous functions are dense in $L^p[a,b]$ for $1 \leq p < \infty$, but are NOT dense in $L^\infty[a,b]$.

One may ask why L^∞ is a subset of L^p but still not dense, it's because the norms are different. Other norms are "simpler to converge" compared to this one?

Remark 7 To be dense in $L^p[a,b]$, it means that for every $f \in L^p$, every $\epsilon > 0$, there exists some $g \in C[a,b]$ so that $\|f - g\|_p$ is less than ϵ.
Consider the second Littlewood's three principles

1. Every measurable set is 'nearly' a finite union of intervals.
2. Every measurable function is 'nearly' continuous.
3. Every pointwise convergent sequence of measurable functions is 'nearly' uniformly convergent. $\square$

Theorem 3.2 (Egoroff's Theorem):
Assume that $m(E) < \infty$. Let (f_n) be a sequence of measurable functions on E that covnerges pointwise on E to the real valued function f. For each $\epsilon > 0$, there is a closed set $F \subseteq E$ for which $(f_n) \to f$ uniformly on F, $m(E \setminus F) < \epsilon$.

Remark 8 What Egoroff says is that if you have a p.w. convergent sequence of functions, and an ϵ, then you can find a subset that is not more than ϵ different in size to the original set, but function converges uniformly there. $\square$

Remark 9 Keep in mind: if some property is true for continuous functions then they might also be true for L^p functions. $\square$

Corollary 3.3:
Polynomials are dense in $L^p[0,1]$ for $1 \leq p < \infty$.

Remark 10 $L^p[0,1]$ is separable for $1 \leq p < \infty$ but $L^\infty[0,1]$ is not. $\square$

Theorem 3.4 (Fubini's Theorem):
Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $f^{-1}(U)$ is a Borel set in $\mathbb{R}^2$ for all open set $U \subseteq \mathbb{R}$. If

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)dx| \right) dy < \infty$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)dx| \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)dy| \right) dx$$