1 LF

1.1 Basic Theory

Lecture 1

- Def: Absolute values, $p-$adic absolute value
- Lem: $p-$adic abs value is an abs value
- Def: equivalent absolute values, place
- Prop: 3 **equivalent conditions for equivalent absolute values**
- Def: non-archimedean absolute value
- Lem: all triangles are isosceles
- Lem: a sequence's condition to be cauchy in nonarch absolute value
- Def: $p-$adic numbers

Lecture 2

- Lem: 4 funny properties for non-archimedean valued fields

1.2 Valuation Rings

- Def: valuation, the $p-$adic valuation, switching from valuation to absolute value
- Def: the $t-$adic valuation on Formal Laurent series
- Def: the valuation ring w.r.t. to a valuation
- Prop: **three properties of $\mathcal{O}_k$ (open subring, unit, its open ideals)**
- Prop: **$\mathfrak{m}$ is a max ideal of $\mathcal{O}_k$, and def of the residue field**
- Cor: $\mathfrak{m}$ is the unique max ideal hence $\mathcal{O}_k$ is a local ring.
- Def: discrete valuation
- Def: Uniformizer
- **Lem: 4 equivalent conditions for v to be discrete. (a condition on v, two conditions on $\mathcal{O}_k$, one on $\mathfrak{m}$)**

Lecture 3

- Def: DVR
- Lem: (K + discrete valuation) $\to$ DVR. DVR $\to K \to \mathcal{O}_k$
- Def: Ring of $p-$adic integers. What is its residue field? max ideal? and all ideals?
- Prop: Relationship between $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$
- Def: Inverse limits
- Def: Projective map Θ_n in the inverse limits

- Prop: Universal property from a S/G/R to an inverse limit
- Def: $I-$adic completion, $I-$adic complete
- Prop: When is $\mathcal{O}_K$ π-adically complete? Also, every $x \in \mathcal{O}_K$ can be written as a $\sum$.
- Cor: Every $x \in K$ can be written as as a $\sum$. Conversely, $\sum$ gives K.

Lecture 4

- Cor: $\mathbb{Z}_p$ is isomorphic to... All $x \in \mathbb{Q}_p$ can be written as...

1.3 Complete Valued Fields

- **Thm: Hensel's Lemma**
- Cor: lifting root consequence of Hensel's lemma
- **Cor: Structure of** $(\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^2$
- **Thm: Hensel's Lemma version 2**

Lecture 5

- Cor: of 2nd version of Hensel's lemma on bounds of coefficients of polynomial.
- Defn: perfect rings, perfect fields
- **Thm: Teichmuller lift theorem. When is** $[-]$ **a homomorphism?**
- Lem: $x, y \in \mathcal{O}_k, k \geq 1$, what does $x \equiv y \mod \pi^k$ imply?
- Lem: In CDVF, if residue fields $\subseteq \overline{\mathbb{F}_p}$, then what are some roots of unity? unfamiliar
- **Thm: A CDVF with** $\operatorname{char} K = p$ **and** k **perfect, what is** K**? unfamiliar**

Lecture 6

- **Thm (big): Given a CDVF** F**, and** L/F **finite extension of degree** n**. Then** $|\cdot|$ **extends uniquely to an absolute value on** L**,** $|\cdot|_L$**, defined by** $|y|_L = |N_{L/K}(y)|^{1/n}$**.**

  In addition, L is complete w.r.t. $|L|$.
- Def: $N_{L/K}(y)$
- Def: If $(K, |\cdot|)$ is an non-archimedean field. Then you can use it to define a norm on a v.s. of K
- Def: equivalent norms
- Def: Sup norm on a vector space that arises from abs value on its field.
- Prop: Any finite dimensional vector space over non-arch complete fields can only have one equivalence class of norm.

  given $(K, |\cdot|)$, complete, non-arch, and V a f.d.v.s. over K. Then, any two norms on V are equivalent. Also,, V is complete w.r.t. any norm.
- Def: $R \subseteq S$ be rings. Then define integral, integral closure, and integrally closed
- Prop: $R^{int(S)}$ is a subring of S and it is integrally closed.

- Lem: Let $(K, |\cdot|)$ be non-arch valued field. Then $\mathcal{O}_K$ is integrally closed in K.

Lecture 7

- Lem: $\mathcal{O}_k^{int(L)} = \mathcal{O}_L$
- Thm: the big theorem proof
- Prop: Uniqueness of extension $|\cdot|_L$
- Cor: $(K, |\cdot|)$ a CDVF, non-arch, discretely valued. Then
  - L is discrete w.r.t. $|\cdot|_L$
  - $\mathcal{O}_L$ is integral closure of $\mathcal{O}_K$ in L.
- Cor: Let $\bar{K}/K$ be the algebraic closure of K. then $|\cdot|$ extends uniquely to an absolute value on $\overline{K}$.
- Prop: downstairs is simple extension implies upstairs is simple extension.

  Conditions: fields are CDVR, finite extension. Then $\mathcal{O}_K$ compact and k_L/k is finite and separable, implies $\mathcal{O}_K[\alpha] = \mathcal{O}_L$.

Lecture 8

- Def: Let $(K, |\cdot|)$ be a valued field. Then K is local if it's ...
- **Prop: If $(K, |\cdot|)$ is an non-archimedean complete valued field. Then you have 3 equivalent conditions.** (i.e. K locally compact, O_k compact, $\mathcal{O}_k/m$ is finite.)
- Def: Profinite topology
- Prop: Let K be a non-archimedean local field. Recall that $\mathcal{O}_K \simeq \varprojlim_n \mathcal{O}_k/\pi^k$ is an isomorphism. It's actually an isomorphism of topological spaces.
- Lem: if K is a non-archimedean local field, if L/K is finite, then L is also local.
- Def: non-arch valued field $(K, |\cdot|)$ has equal char if... has mixed char if...
- Thm: If K is a non-arch local field of equal char, then $K \cong$ ...
- Lem: Absolute value on K is non-archimedean $\iff$ $|n|$ is bounded $\forall n \in \mathbb{Z}$.

Lecture 9

- **Ostrowski's theorem**
- **Thm: Non-arch local field of mixed character, then is finite extension of $\mathbb{Q}_p$ for some p.**
- **Classification of local fields**

1.4 Global fields

- **Defn: Global field**
- Lemma: Let L/K be finite Galois. Then elements in L has the same image as its image under galois elements
- **Lem: Krasner's lemma**
- **Prop: nearby polynomials define same extensions**

Lecture 10

- **Thm: Classification of local fields**

1.5 Dedekind domains

- **Defn: dedekind domain**
- **Thm: A ring is DVR $\iff$ it's DDK and has exactly one nonzero prime ideal**
- Lem: in an Noetherian ring R, let I be an ideal. Then there exists nonzero prime ideals such that $p_1 \dots p_n \subseteq I$.
- Lem: Let R be an ID. Let R be integrally closed. let $I \subseteq R$ be f.g. ideal. If $x \in K, xI \subseteq I$ then $x \in R$.
- Def: Localisation
- Cor: DDK domains localized is DVR

Lecture 11

- **Thm: In DDK, ideals factors**
- Prop: Two properties of localisation

1.6 Dedkind domains and extensions: Now study L/K finite separable field extension and how they relate.

- Given a field extension L/K and a linear map $\cdot x$, what is the trace? What can you write it as?
- **Defn: trace form**
- Lemma: Trace form is non-degenerate
- Lemma: Integral closure of DDK is DDK. Given a DDK $\mathcal{O}_k$. Let $K = \text{Frac}(\mathcal{O}_K)$. Let L/K be a finite separable extension. Then let $\mathcal{O}_L$ be $\mathcal{O}_K$'s int closure in L. Then $\mathcal{O}_L$ is DDK. Proof skipped.
- Cor: Ring of integers in a number field is DDK.
- **Def: p-adic absolute value on a number field**

Lecture 12

Setting: L/K finite separable. $K = \text{Frac}(\mathcal{O}_K)$. $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L. $\mathcal{O}_K$ be DDK.

- **Defn:** V_p
- Lem: $0 \neq x \in \mathcal{O}_K$ then $(x) = \prod_p p^{v_p(x)}$
- **Thm: Absolute values of L extending $|\cdot|_{\mathfrak{p}}$**
- Cor: generalisation of Ostrowski: Apply Ostrowski on number fields

1.7 Completion of dedekind domains

- **Lem:** $\pi_{\mathcal{P}} : L \otimes_K K_{\mathfrak{p}} \to L_{\mathcal{P}}, (l, k) \mapsto lk$ **is subjective.** $[L_{\mathcal{P}} : K_{\mathfrak{p}}] \leq [L : K]$
- **Thm:** $L \otimes_K K_{\mathfrak{p}} \to \prod_{\mathcal{P}|\mathfrak{p}} L_{\mathcal{P}}$ **is an iso**

Lecture 13

- **Cor:** $x \in L$, **then** $N_{L/K}(x) = \prod_{\mathcal{P}|\mathfrak{p}} N_{L_{\mathcal{P}}/K_{\mathfrak{p}}}(x)$
- **Defn: Ramification index and ramifies. Residue class degree**

- **Thm:** $\sum_i e_i f_i = [L : K]$
- **Prop:** L/K **Gal then** $\mathrm{Gal}(L/K)$ **on** $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ **is transitive**
- **Cor:** L/K **gal then** $n = efr$
- Cor: some constants for extensions of DVF
- **Defn: Decomposition group**
- Prop: any $\mathcal{P}, \mathcal{P}'$ are conjugates.

Lecture 14

- **Prop: If** L/K **is Galois,** $\mathcal{P} \mid \mathfrak{p}$ **is prime ideal of** $\mathcal{O}_L$
  - **Then** $L_{\mathcal{P}}/K_{\mathfrak{p}}$ **is galois**
  - **The map** $res : \mathrm{Gal}(L_{\mathcal{P}}/K_{\mathfrak{p}}) \to \mathrm{Gal}(L/K)$ **is injective and has image the decomposition group.**

1.8 Different and the discriminant

- **Defn:** $\Delta : L^n \to K$
- Lemma: trace form nondegenerate $\iff$ R can be written as a product of finite extensions of K.
- **Thm:** $0 \neq \mathfrak{p} \subseteq \mathcal{O}_K$ **a prime ideal. Then** $\mathfrak{p}$ **is ramified** $\iff \ldots$, **and unramified** $\iff \ldots$.
- Defn: The discriminant ideal: $d_{L/K}$
- Cor: $\mathfrak{p}$ ramifies in $L \iff \mathfrak{p} \mid d_{L/K}$
- Defn: the inverse different: $D_{L/K}^{-1}$
- Lem: the inverse different is a fractional ideal. Proof skipped but dont plan on seeing it anyways
- Defn: the different ideal
- Defn: $N_{L/K} : I_L \to I_K, \mathcal{P} \mapsto \mathfrak{p}^f$ is a group homomorphism. I_K, I_L denotes the fractional ideals in the respective fields

Lecture 15

- **Prop:** $L^\times, K^\times, I_L, I_K$ **commutes with two different definitions of** $N_{L/K}$
- Thm: $N_{L/K}(D_{L/K}) = d_{L/K}$ Proof gladly omitted
- Thm: $D_{L/K} = (g'(\alpha))$ Proof gladly omitted
- Thm: $D_{L/K} = \prod_{\mathcal{P}} D_{L_{\mathcal{P}}/K_{\mathfrak{p}}}$ Proof very happily omitted
- Cor: $d_{L/K} = \prod_{\mathcal{P}} d_{L_{\mathcal{P}}/K_{\mathfrak{p}}}$ Proof very happily omitted

1.9 Unramified and totally ramified extensions of local fields

- Cor: $[L : K] = e_{L/K} f_{L/K}$
- Lemma: Tower law for e, f
- Defn: unramified, ramified, and totally ramified extensions

Lecture 16

1.10 Unramified and Totally Ramified extensions of local fields

- Cor: $[L : K] = e_{L/K} f_{L/K}$
- Lemma: Tower law for non-archimedean extensions $M/L/K$
- Def: Unramified/ramified/totally ramified extension.
- **Thm: Finite separable extension of local fields L/K splits into an unramified one and a totally ramified one.**
- Thm: Unramified extensions of non-archimedean local fields are easy to understand! What can you say about the map $\mathrm{Gal}(K_0/K) \to \mathrm{Gal}(k_0/k)$? Not sure about the Frobenius generator part. $\mathrm{Frob}_{L/K}(x) = x^q \mod m_L$
- Cor: L/K finite separable ext of LF, then $res : \mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k)$ is surjective.
- Def: Inertia subgroup. What is its order?
- Def: Eisenstein polynomial
- Fact: What can you say about Eisenstein polynomials?
- **Thm: Relationship between totally ramified and eisenstein polynomial**
- *Unramified parts controlled by extension by roots of a cyclotomic polynomial. Ramified parts controlled by Eisenstein Polynomials*

1.11 Structure of Units

Lecture 17

- Defn: Absolute Ramification Index
- Now switch to work in $[K : \mathbb{Q}_p] < \infty$. Following must work in mixed char otherwise factorial breaks!!!
- Prop: Under some circumstances, exp converges and induces isomorphism between the additive and multiplicative structures of units. What is the special r?

  Two important lemmas in this proof:

  - $v_p(n!) = \frac{n - S_p(n)}{p-1}$ where $S_p(n)$ is the sum of p-adic digits of n.
  - $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$
- Defn: The s^{th} unit group $U_K^{(s)}$
- Remark: The unit group filtration
- **Prop: What are the quotients of filtration for unit groups?**
- Thm: $[K : \mathbb{Q}_p]$ has a finite index subgroup of $\mathcal{O}_K^\times$ that is isomorphic to $(\mathcal{O}_K, +)$
- Example of unit groups, in $\mathbb{Z}_p$ or $\mathbb{Z}_2$

1.12 Higher ramification group

- **Defn: Higher ramification groups** G_s, write G_s as a normal subgroup
- Example: of higher ramificaiton groups
- **Thm: Three properties about higher ramification group**
  1. For $s \geq 1$, G_s can be defined equivalently with...
  2. $\bigcap G_s$
  3. G_{i+1}/G_i

Lecture 18

- Cor: For a finite Galois extension of local fields L/K, have $\mathrm{Gal}(L/K)$ is solvable.
- Def: Wild Inertia group, then tame quotient.
- Def: Tamely ramified vs wildly ramified. Tamely ramified if the wild inertia group is trivial. Wildly ramified otherwise.
- *So, all three* G_{-1}, G_0, G_1 all have names. The Gal, the inertia, the wild inertia
- Thm: A theory relating $D_{L/K}$ with whether the extension is tamely ramified. Skipped because this seem not as essential and kinda hard
- Cor: L/K extension of number fields. If $P \mid p$, then $e(P \mid p) > 1 \iff P \mid D_{L/K}$. Skipped because this seem not as essential and kinda hard
- Example: computing higher ram groups

2 Local Class Field Theory

2.1 Infinite Galois Theory

- Note: Familiarity with basic Galois theory: separable, normal, Galois extension, Galois correspondence
- *To generalize the above on infinite galois extension, need some theory on inverse limits*
- Defn: Directed set. What are two examples?
- Defn: Inverse system of groups, translation homomorphism, inverse limits.
- **Prop: Putting inverse system on the Galois group**
- Question: what is $\mathrm{Gal}(L/K)$ written in terms of the inverse system? When L/K Galois is not necessarily finite. Proof: Skipped

Lecture 19

- Defn: Profinite topology defined on an inverse system of directed set of groups
- Example: inverse system of Galois group on $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ and a commutative diagram explaining it.
- Thm: Fundamental Thm of Galois Theory (infinite galois extension). Closed subgroup vs open subgroup?

2.2 Weil Group

Let K be local fields and L/K a separable algebraic extension

- Defn: (In the case of infinite extension) unramified extension and totally ramified extension
- Thm: Let L/K be unramified. Then L/K is Galois and what can you say about $\mathrm{Gal}(L/K)$? (Proof includes a comm diag)
- Notation: $Fr_{k_{L/k}}$
- **Defn: The weil group**
- Example: a commutative diagram involving the Weil group
- Def: Topology on the Weil group

Lecture 20

- **Prop: Relationship between $W(L/K)$ and $\mathrm{Gal}(L/K)$. Three statements saying that we don't lose information by just looking at $W(L/K)$.**
- **Example: Commutative diagram of exact rows showing Gal and W.**
- Question: under what condition is $W(L/K) = \mathrm{Gal}(L/K)$?
- $I_{L/K} = \mathrm{Gal}(K_0/K)$

2.3 Statements of local class field theory

- Defn: Abelian extension, facts about them
- **Defn: K^{ur} and K^{ab}**
- Thm: Local artin reciprocity: Exists map $K^\times \cong W(K^{ab}/K)$
- Section: Properties of artin map and relationship between $e_{L/K}$ and $N_{L/K}$: This whole part gladly skipped.
- Thm: Local Kronecker Weber $\mathbb{Q}_p^{ab} = \mathbb{Q}_p^{ur}\mathbb{Q}_p(\zeta_{p^\infty})$

Lecture 21

- Construction of Art K, motivation of it, etc
- Commutative diagram: the commutative diagram with $K_{\pi,n}/K$, how the construction of $K_{\pi,n}$ need Lubin tate theory. If we set $K_{\pi,\infty}$ then $K^{ab} = K^{un}K_{\pi,\infty}$, commutative diagrams comparing $\mathbb{Q}_p^{ab}$ and K_p^{ab}

2.4 Lubin Tate Theory

- **Defn: $1-$dimensional formal group law**
- **Lem: two properties of formal group law: $F(0,X) = X$ and exists inverse**
- Prop: in non-archimedean valued field, it converges in resdieu field
- Prop: in residue field, formal group law gives you a group
- **Def: homomorphisms and isomorphisms of formal group laws**
- Prop: exp, log are isomorphism of formal group laws
- Lemma: $\mathrm{End}_R(F)$ is a ring with addition $f +_F g(x) = F(f(x), g(x))$ and multiplication given by composition

2.5 Lubin Tate formal group

- **Defn: Formal $\mathcal{O}_K$ module.**
- **Defn: hom or iso of formal $\mathcal{O}_K$ module**
- **Defn: Lubin tate series**

Lecture 22

- **Thm: Big theorem in Lubin tate module**

  Let $f(x)$ be a lubin tate series for π.

  Then there are three properties for $f(x)$:

  - exists unique group law F_f over $\mathcal{O}_K$ so that $f \in \text{End}(F_f)$
  - Exists ringhomomorphism
    $$[-]_f : \mathcal{O}_K \to \text{End}(F_f)$$
    that is, F_f is a formal ok module over ok
  - If $g(x)$ is another formal lubin tate series for π then $F_f \simeq F_g$ as formal $\mathcal{O}_K$ modules

- **Defn: Lubin tate formal group law**
- **Lemma: Key lemma to prove the big theorem**
- **Proof: of big theorem**

Lecture 23

- **Lemma: Let F be a formal $\mathcal{O}_K$ module. Then m, the max ideal of algebraic closure of K is a $\mathcal{O}_K$ module**
- **Defn: π^n torsion group**
- Defn: $f(x), f_n(x), h_n(x)$ in the context of the π^n torsion group
- Prop: $h_n(x)$ is a seprable eisenstein of degree $q^{n-1}(q-1)$ Proof happily omitted
- Prop: $\mu_{f,n}$ is a free module of rank 1. If g is another lubin tate series for π then $\mu_{f,n} \simeq \mu_{g,n}$Proof happily omitted
- **Def: Lubin tate extension $K_{\pi,n}$**
- Prop: $K_{\pi,n}$ totally ramified and Galois extensiono f degree $q^{n-1}(q-1)$

Lecture 24

- **Thm: $\Phi_n : \text{Gal}(K_{\pi,n}/K \to (\mathcal{O}_k/\pi^n\mathcal{O}_K)^\times$ is an isomorphism**
- Generalized kronercker weber: $K^{ab} = K_{\pi,\infty}K^{ur}$