1 Commutative Alg Active Recall

1.1 Introduction

- Defn: algebraic set, vanishing locus
- Prop: $S \subseteq k[T_1, \ldots, T_n]$. Let $I = \langle S \rangle$, then $V(S) = V(I)$

1.2 Noetherian rings and Hilbert's basis theorem

- Idea: Motivation for Hilbert's basis theorem
- **Prop: Three equivalent definitions of Noetherian rings**
- Idea: Chain of inclusions of different types of integral domains
- Lem: $\phi : A \to B$ is a ring hom. Then if A is Noetherian, so is $\phi(A)$.
- Prop: preimages of ideals are ideals.
- Def: let A, B be rings. Then a A is a $B-$algebra if...
- Def: B-algebra homomorphism, B-subalgebra
- Def: B-subalgebra generated by S'. Both as an intersection and written explicitly.
- Def: finitely generated B algebra. Both in terms of literal sense and a quotient sense.
- Idea: The correspondence finitely generated B algebras and $\iff$ quotients of $B[T_1, \ldots, T_m]$
- **Thm: Hilbert's basis theorem**
- Thm: If $S \subset k[T_1, \ldots, T_n]$ then there exists a finite $S_0 \subset S$ such that $\langle S \rangle = \langle S_0 \rangle$.

1.3 The noether normalization theorem

- Defn: Let A be a B- algebra. Then A is finite over B if...
- Idea: $k[T, T^{-1}]$ is not finite over k or $k[T]$ but it is over $k[T - T^{-1}]$
- Defn: Let A be a $B-$algebra. Then A is integral over B if...
- Idea: compare integral extension and algebraic extension
- Lem: Let C be a $n \times n$ matrix over a ring A. If $V \in A^n$, and $Cv = [0, \ldots, 0]^T$, then $\det(C)v = [0, \ldots, 0]^T$.
- Lem: Let $B \subset A$ be rings. Then $x \in A$ is integral over B if there is a $B[x]$ module M of A such that
  - M is faithful as a $B[x]$ module
  - M is finitely generated as a $B-$module.
- Idea: if M is a module over A, then if $1 \in M$, then it's faithful.
- **Prop: Three equivalent properties of A being a finite B-algebra.** Why does $\phi(B)$ algebraic imply $B-$algebraic?
- Defn: Let A be a k-algebra over field k. Then algebraic independence means... The literal definition and definition as a map in the polynomial ring.
- **Thm: Noether's Normalization Theorem**
- Lem: Schwartz–Zippel Lemma

1.4 Hilbert's Nullstellensatz

- Idea: bijection between $k^n \rightleftharpoons \hom_{k-\mathrm{alg}}(k[T_1,\dots,T_n],k)$
- Prop: $\ker(f_x) = (T_1 - x_1, \dots, T_n - x_n)$
- Prop: $(T_1 - x_1, \dots, T_n - x_n)$ is a maximal ideal
- Prop: The map $k^n \to \operatorname{mspec} k[T_1,\dots,T_n], (x_1,\dots,x_n) \mapsto (T_1 - x_1, \dots, T_n - x_n)$ is injective. (Note: if k alg-closed, then it is surjective. Otherwise, find an example of non-alg closed k such that the map is not surjective.)
- Rem: Intuition for Hilbert's strong NSZ: $I(V(T^2)) = (T)$, so $I(V(\bullet))$ is like taking roots.
- Def: radical of an ideal. $\sqrt{I}$
- Prop: Let $A \subset B$ be integral domains. B is integral over A. Then $A \cap B^\times = A^\times$
- Lem: Let $A \subset B$ be integral domains. B integral over A. Then B is a field $\iff$ A is a field.
- **Prop: Zariski's Lemma: let $k \subset K$ be fields. If K is finitely generated as $k-$algebra, then K is finite as a k algebra. (i.e. $\dim_k K < \infty$.)**
- **Thm: The weak NSZ: For a field k, a proper ideal $\mathfrak{a}$ of k, there is a field extension L of k and $x \in L^n$ such that $f(x) = 0, \forall x \in \mathfrak{a}$. (if k is algebraically closed, $L = k$.)**
- Cor: if k is algebraically closed, there is a bijection $k^n \to \operatorname{mspec} k[T_1,\dots,T_n]$.
- Def: $V_{k^{al}}(\mathfrak{a}) = \{x \in k^{al} \mid f(x) = 0, \forall f \in \mathfrak{a}\}$
- **Thm: The strong NSZ:** $I(V_{k^{al}}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$
- Prop: The statement representing the harder direction for NSZ
- Rem: note that Strong NSZ shows $I(V_{k^{al}}(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$, where $I(V_{k^{al}}(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$ and $V(I(X)) = X$ are easier claims that only require elementary set theory.
- Rem: bijections between radical ideals of $k[T_1,\dots,T_n]$ and algebraic subsets of k^n.

1.5 The Zariski topologies on Kn and Spec(A)

1.6 The Space Spec(A)

- Let I be an ideal. What is $I(V)$? and what are the closed subsets of $Spec(R)$?
- **Thm: The zariski topology of $Spec(R)$ is indeed a topology.**
- What are basis of zariski topology?

1.7 Localization

- Def: multiplicative subset
- **Def:**$S^{-1}A$
- **Theorem:** addition and multiplication are well defined in $S^{-1}A$
- **Def:** inclusion map $i_A : A \to S^{-1}A$
- **Thm: Universal property for $S^{-1}A$**

- Lemma: if $0 \in S$ then $S^{-1}A$ is trivial
- Def: The ring A_h
- **Thm:** $A[T]/(1-hT) \cong A_h$
- Def: contraction and extension of an ideal
- Lemma: If $A \hookrightarrow B$ then $\mathfrak{b}^C = A \cap \mathfrak{b}$.
- Lemma: Surjective homomorphisms sends ideal to ideals
- Def: contracted ideals and extended ideals
- **Prop:** $\mathfrak{a}^{ec}, \mathfrak{b}^{ce}, \mathfrak{a}^{ece}, \mathfrak{b}^{cec}$
- Remark: A map between contracted ideals and extended ideals
- Prop: If $\mathfrak{b}$ is a prime ideal of B then $\mathfrak{b}$ is prime $\iff$ $\mathfrak{b}^C$ is prime.
- Remark: Contracted and extended ideals in the context of localisation
- **Prop:** Consider S, A localisation with extension/contraction of ideals.
  - $\mathfrak{b}^{ce} = \mathfrak{b}$
  - **Bijection of ideals: ones that avoids and ones......**
  - How to prove? One may need $S^{-1}A/\mathfrak{p}^e \cong \overline{S}^{-1}(A/\mathfrak{p})$
- Defn: $A_\mathfrak{p}$
- Prop: When let $S = A \setminus \mathfrak{p}$, the above statement becomes... What is the maximal ideal?

1.8 Going up and going down

- Prop: $A \subset B$ integral. $\mathfrak{b}$ an ideal of B. Then $A/\mathfrak{b} \cap A \hookrightarrow B/\mathfrak{b}$ is integral.
- Cor: $A \hookrightarrow B$ be integral extensions. Let $\mathfrak{q}$ be a prime ideal of B. then $\mathfrak{q} \cap A$ is a max ideal in A $\iff$ $\mathfrak{q}$ is a max ideal in B.
- Remark: If $A \subseteq B$, then explain $S^{-1}A \to S^{-1}B$ as a homomorphism
- Prop: If $A \subset B$ integral, then $S^{-1}A \subset S^{-1}B$ is integral
- **Prop: Incomparability**
- **Prop: Lying over**
- **Prop: Going up**
- List of things to show going down
- Example: not-integral extension, going up fails
  - Defn: integral closure: if $A \subset B$, what is the integral closure of A. For A an ID standalone, what is integral closure of A?
  - Thm: integral closure is a subring
  - Defn: Integrally closed
  - Prop: Every UFD is integrally closed.

  - **Prop: A an integrally closed ID. E a finite extension of A. Then $\alpha \in E$ is integral over A iff ...**
  - Def: element integral over ideal
  - Prop: Equivalent condition for being integral over an ideal
  - Prop: Integral closure of an ideal
  - Prop: a prime ideal is the contraction of a prime ideal $\iff p^{ec} = p$
  - **Thm: going down** This is a big proof. It's skippable.

1.9 Dimension theory for finitely generated algebras over a field

- **Def: Height of an ideal**
- **Def: Krull dimension of a ring**
- Prop: Integral domain A is a field $\iff \dim(A) = 0$
- Prop: What is the dimension of a PID?
- **Prop: 3 equivalent definitions of transcendental basis**
- Prop: Three properties of transcendental basis
- Defn: Trdeg
- **Thm: if A is a f.g. K-algebra then** $trdeg_K(A) = \dim(A)$
- Defn: Localize at an element
- **Prop: Let R be a ring. $n \geq 0$. Then,** $\dim R \leq n \iff \dim R_x \leq n-1, \forall x \in R$
- Three properties about localisation at an element.
- **Prop: A an ID, K a subfield of A, then** $trdeg_K(A) \geq dim(A)$
- Prop: Let $A \subset B$ be integral extension of rings. Then
  - $\dim A = \dim B$
  - If A, B are ID, K-alg, $K-$field, A a $K-$subring of B, then $trdeg_K(A) = trdeg_K(B)$
- **Prop: Trdeg(A)=**$\dim A$

1.10 Nakayama's lemma and applications

- **Thm: Nakayama's lemma**
- **Thm: Krull's intersection Theorem**

1.11 Artinian rings

- **Def: Artinian Rings**
- Prop: Artinian ring if and only if every chain of ideals...
- Examples of Artinian rings: when is artinian ring a field? $K[T]/\langle T^n \rangle$?, $K[T]$?, $\mathbb{Z}$?
- **Prop: Dimension of non-zero artinian ring**
- Def: Nilradical and jacobson radical
- Prop: In an artinian ring, what is the relationship between its Nil and J?
- **Prop: An artinian ring only hsa finitely many max ideals**
- **Prop: If A is artinian, what can you say about $nil(A)$?**
- **Def: Noetherian modules and artinian modules**
- Prop: If ..., then A is Artinian $\iff$ it is Noetherian Unfamiliar with the proof
- Thm: A is Noetherian $\iff$ it is artinian with dim 0.

1.12 Dimension theory for noetherian rings

- Defn: Exact sequence, SES
- **Defn: Graded rings**
- Prop: In a graded ring, what can you say about A_0 and A_n?
- **Graded A-module**
- Def: homogenous elements in M. What do general elements in M look like?
- Def: homomorphism of graded A modules
- Def: A_+
- Prop: For a ring A, A is noetherian if and only if... A_0... and ...
- **Defn: Additive function**
- Prop: Alternative sum of λ of a LES is 0 given that λ is an additive function
- Def: composition series
- Lemma: let M be a module. Then all of its composition series have common length and any chain can be refined to composition series.
- **Defn: $\ell(M)$**
- Prop: M has finite length $\iff$ M is artinian and noetherian
- **Remark: Setting for Hilbert functions** Why is M_n a finitely generated A_0 algebra?
- **Defn: Poincare series**
- **Thm: Hilbert- Serre**
- Now, assume that λ is positive definite. Define $d(M)$.

- **Prop: $d(M/xM) = d(M) - 1$ if $x \in M$ is not a zero divisor**
- **Prop: existence of Hilbert Polynomial**
- **Defn: Hilbert Polynomial**
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- **Def: Filtrations**
- **Def: $\mathfrak{a}$-filtration, stable $\mathfrak{a}$-filtration**
- **Lemma: Bounded difference** Not too familiar with proof. Come back to it another time.
- **Prop: given an ideal. Mkae a graded ring A^*. Given an $\mathfrak{a}$ filtration, make a graded A^* module.**
- **Prop: A noetherian implies A^* noetherian**
- **Lemma: M^* f.g. A^* module implies M_n stable.** Unfarmiliar!
- **Prop: Artin Rees-Lemma**Unfarmiliar!
- **Def: Associated graded ring and graded $G(A)$ module.**
- **Def: Graded $G(A)$ modules.**
- **Prop: Three properties about associated A modules**
- **Defn: Primaryy ideals**
- Thm: The dimension theorem: Three numbers of A. This is a big theorem so I skipped entirely.
- **Cor: Krull's height theorem**

1.13 Tensor products

- Def: Free A-module over S
- **Def: Tensor Product**
- **Thm: Universal property for tensor product**
- Def: tensors vs pure tensors
- Prop: $(M \otimes N, i_{M\otimes N})$ is the only pair that satiesfies universal property up to iso
- Prop: $\sum_{i=1}^{l} m_i \otimes n_i \neq 0$ if and only if...
- Embedding of a tensor doesn't work
- Prop: If $\sum m_i \otimes n_i = 0$ then there exist f.g. A modules, $M' \subset M, N' \subset N$ such that $\sum m_i \otimes n_i = 0$ in $M' \otimes N'$.
- **Prop: Five natural isomorphisms of tensor products**
- Example: tensor prod of v.s. are v.s. with basis the pure tensors
- **Def: restriction of scalars and extensions of scalars**
- Prop: Show that extension of scalars makes sense

- Examples of extensions of scalars
- Defn: tensor product of algebras and why it makes sense. Two ways to make $B \otimes C$ into an A-algebra.
- Change base field of f.g. algebras
- Defn: tensoring homomorphisms

1.14 Flat modules

- **Prop: If N is a module then the functor $M \to M \otimes N$ is right exact**
- Counter example for the above $M' \to M \to M''$ doesn't work ($\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$)
- **Defn: Flat module**
- Example: Give two examples of flat modules
- Example: Flat modules are torsion free
- **Defn: Torsion free A module**
- **Defn: Free resolution and tor functor**
- Example of Tor given $x, A, A/(x)$
- Lemma: Ring modulo ideal with Tor
- Prop: Equivalent condition for a module being flat v.s. its ideals