Week 1 lecture 1.

def. algebraic set & vanishing locus.

let K be a field, $A = K[T_1, \cdots T_n]$ let $S \subseteq A$.

Then $V(S) = \{x \in K^n \mid f(x) = 0 \quad \forall f \in S\}$

a set $X \subseteq K^n$ is algebraic if $X = V(S)$ for some $S \subseteq A$.

Prop: $V(S) = V(I)$

let $I = \langle S \rangle$

$= \bigcap \{I \mid I \text{ an ideal of } A, \ S \subseteq I\}$

$= \{ s_1 a_1 + \cdots + s_n a_n, \ n \in \mathbb{Z}, \ s_i \in S, \ a_i \in A\}$

since $S \subseteq I$, $V(S) \supseteq V(I)$ as any point vanishing in all of I must vanish in all of S.

to show $V(I) \supseteq V(S)$ if $p \in K^n$ disappear in all $\langle S \rangle \supseteq S$ then $f(x) = 0 \ \forall f \in S$.

Noetherian Rings & Hilbert's Basis theorems.

Motivation.

let $S \subseteq A = K[T_1, \cdots T_n]$. S is possibly infinite. let $X = V(S)$. does there exist finite $S_0 \subseteq A$ s.t. $X = V(S_0)$? Yes by Hilbert's basis theorem.

Defn Noetherian rings

3 equivalent conditions.

1) every ideal is finitely generated
2) every ascending chain of ideals stabilize.

i.e. $I_1 \subseteq I_2 \subseteq \cdots$ then $\exists n$ s.t. $I_n = I_m \ \forall m > n$.

3) every set of ideal has a maximal element.

Proof :

i) ⇒ ii) let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals then,

$I = \bigcup_{i=1}^{\infty} I_i$ is an ideal of A.

let $I = (a_1, \cdots a_n)$. let $n_i \in \mathbb{Z}$ s.t. $a_i \subseteq I_{n_i}$. Then set $m = \max_i \{n_i\}$.

Then each $a_i \in I_m$ so chain stabilizes at I_m.

ii) ⇒ iii)

let S be a set of ideals. If S has no maximal element,

have ideals $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots$ contradicting ii)

iii) → i)

let $I \subseteq A$ be an ideal. consider $A = \{I'$ an f.g. ideal, $I' \subseteq I\}$.

By iii), exists $b \in A$ a maximal element. Say $b = (b_1, \cdots b_n)$ $b_i \in A$

Recall that b is f.g.

Claim $b = I$.

$b \subseteq I$ ✓

$I \subseteq b$ Suppose not. $\exists x \in I \setminus b$. But then $(b_1, \cdots b_n, x)$ is another f.g. ideal contained in I. #.

Proof Scheme :

i) ⇒ ii) take union of chain

ii) ⇒ iii) contrapositive + AOC

iii) ⇒ i) consider all f.g. sub ideals.

example : 1) ideals in K is either (0) or (1)

2) $K[T_1, T_2, T_3, \cdots]$

non-noetherian ring b/c $k[T_1] \subseteq k[T_2, T_3] \subseteq \cdots$

3) PIDs are Noetherian again

field ED PID UFD ? ID.

Noetherian

lem: $\varphi: A \to B$ a ring hom. A is Noetherian then $\varphi(A)$ is Noetherian.

is $\varphi(A)$ a ring? yes!

Proof. let $C \subseteq \varphi(A)$ be an ideal of $\varphi(A)$.

$\varphi^{-1}(C)$ is an ideal of A.

if $x \in \varphi^{-1}(C)$ $y \in \varphi^{-1}(C)$ then $\varphi(x), \varphi(y) \in C$ so $\varphi(x+y) \in C$, $x+y \in \varphi^{-1}(C)$

if $x \in \varphi^{-1}(C)$ and $y \in A$ $\varphi(x) \in C$ $\varphi(x)\varphi(y) \in C$ $\varphi(xy) \in C$ $xy \in \varphi^{-1}(C)$

then, A is Noetherian. So $\varphi^{-1}(C) = (a_1, a_2, \cdots a_n)$

then $C = (\varphi(a_1), \cdots \varphi(a_n))$

Prop If $C \subseteq B$ is an ideal, $\varphi^{-1}(C)$ is an ideal of A.

def. an algebra A over a ring B is a ring equipped with the structure homomorphism $\varphi: B \to A$. Think of B as ring of scalars acting on A.

i.e. $b \in B$, $x \in A$, write $bx = \varphi(b)x$.

φ: "lift" scalar into an element in ring that can act on.

Two examples of B-algebras:

1) $A = k[x_1, \cdots x_n]$ is a k algebra. $\varphi: k \mapsto k$ ← constant polynomial.

2) A any ring is a $\mathbb{Z}$-algebra. $\varphi: \mathbb{Z} \to A$, $\varphi(1) = 1$

note φ is also unique as it must send 1 to 1.

def B-algebra homomorphism

$f: A \to A'$ is a B-algebra homomorphism if $f(bx) = bf(x)$ $\forall b \in B$ $x \in A$.

If $B \subset A$ then A is a B-algebra by inclusion also a B-algebra hom.

def B-subalgebra

let A be a B-alg. Then, $A' \subset A$, a subring of A is an B-subalg

if $bx \in A'$ $\forall$ $x \in A'$. $\forall b \in B$

def: B-subalg generated by S'

let $S \subset A$ be finite, then the B-sub alg of A generated by S is

{ intersection of all B-sub algebra containg S

or consist of elements of form $f(s_1, \cdots, s_m)$ where $f \in B[T_1, \cdots T_m]$. $s_i \in S$.

def. finitely generated

A, a B-alg is f.g. if A is a B-subalgebra of A generated by a finite set.

Prop. finitely generated B-algebra as a quotient

if A is a f.g. B-alg, then let $S = \{s_1, \cdots s_m\}$ be the set that generates it.

then, have a surjective B-alg hom $g: B[T_1, \cdots T_m] \to A$.

$$p \mapsto p(s_1, s_2, \cdots s_m).$$

so A is a quotient of $B[T_1, \cdots T_m]$

also, every quotient of $B[T_1, \cdots T_m]/I$ is generated by the finite set $(T_1+I, \cdots, T_m+I)$.

Therefore, we have a correspondnce btwn

" f.g. B-alg " and " quotients of $B[T_1, \cdots T_m]$ "

Thm: Hilbert's basis theorem

let A be a f.g. B-alg. If B is Noetherian, so is A.

Proof: Step 1: reduce to problem about $B[T]$.

A is a f.g. B-alg so A is a quotient of $B[T_1, \cdots T_m]$.

$A \cong B[T_1, \cdots T_n]/I$. So have hom $B[T_1, \cdots T_n] \to A$ So by prev lemma (image of Noe is noe) suffice to show $B[T_1, \cdots T_n]$ Noe. But $B[T_1, \cdots T_n] \cong B[T_1, \cdots T_{n-1}][T_n]$ as B-algebras.

So it suffice to show $B[T]$ is Noetherian then use induction.

Step 2: make $a(i) \subseteq B$.

let a be an ideal of $B[T]$.

write $a(i) = \{c_0 \mid c_0 T^i + \cdots + c_i T^0 \in a, \ c_0, \cdots c_i \in B\}$. i.e. $a(i)$ is the set of leading coefficients of degree i polynomials in a.

then,

① $a(i) \subset a(i+1)$

let $k \in a(i)$ so have poly $p = kT^i + \cdots \in a$

but $pT \in a$ so $\underbrace{kT^{i+1} + \cdots}_{i+1 \text{ deg pol in } a} \in a.$

So $a(i) \subset a(i+1)$

② $a(i)$ is an ideal of B.

let $k \in a(i)$ $b \in B$, then have poly $p = kT^i + \cdots \in a$

Step 3. Stabilize in B.

but $bp = (bk)T^i + \cdots \in a$ too so $bk \in a(i)$.

Since B noetherian, $a(0) \subseteq a(1) \subseteq \cdots$ stabilizes. Say $\exists m$ s.t. $a(m+j) = a(m)$ $\forall j \geq 0$.

Say each ideal $a(i)$ is generated by some finite subset $\{b_{i,1}, \cdots b_{i,n_i}\}$ of B,

Step 4. construct b.

By defn of $a(i)$, $\exists f_{ij} \in a$ s.t. $f_{ij} = b_{ij}T^i + \underbrace{\{\cdots\}}_{\text{poly of deg} < i}$

let b be the ideal generated by $\{f_{ij} \mid 0 \leq i \leq m, 1 \leq j \leq n_i\}$

We claim $a = b$.

$b \subset a$ by construction

$a \subset b$. first claim $a(i) = b(i)$

if $k \in a(i)$ then $a(i) \subseteq B$ f.g. so in $b(i)$

Suppose ※ $a \setminus b \neq \emptyset$ if $k \in b(i)$ by inclusion ✓.

let $f \in a \setminus b$ of least degree in $a \setminus b$. let $i = \deg f$. $b(i) = a(i)$ So l.c. of $f \in a$ must be the l.c. of $g \in b \subseteq a$ then $\deg(f-g) < i$.

since f, g both in a, $f-g \in a$ but minimality implies $f-g \in b$.

So $f = (f-g) + g \in b$. ※

Proof Scheme

↳ using quotient of $B[T_1, \cdots T_n]$, and induction, and $\mathrm{Im}(\mathrm{Noe}) = \mathrm{Noe}$ to reduce the problem to $B[T]$.

↳ let $a \in B[T]$. construct $a(i)$ s.t. it satisfy acc

↳ construct $b_{i,j}$. Then construct the ideal b.

↳ notice $b(i) = a(i)$, claim $a = b$. if not, use minimality of deg of $a \setminus b$ to cook up a contradiction.

Thm. in $K[T_1, \cdots T_n]$ if S is a set in $K[T_1, \cdots T_n]$, then consider $I(S)$.

then claim $\exists S_0 \subset S$ s.t. $\langle S \rangle = \langle S_0 \rangle$.

Pf: Since K is noetherian, $K[T_1, \cdots T_n]$ is Noetherian, so any ideal is finitely generated by some (S_0).

now show $S_0 \subset S$:

general statement: If an ideal in ring B can be generated by a set T, and by T' seperately, if T' finite, then it can be generated by a finite subset of T.

Proof. $(T) = (T')$

T' finite, Write each $t_i \in T'$ as $t_i = \sum_{j=1}^{n_i} b_{ij} r_{ij}$, each $r_{ij} \in T$.

then let $T_0 = \{ r_{ij} \}$. $(T_0) \subset (T)$

$(T) \subset (T') \subset (T_0)$

Note an B-algebra A can be viewed as a B module by forgetting the multiplication.

defn. let A be a B-algebra. Then A is finite over B if A is f.g. as a B module.

remember: A is finitely generated over B:

$\exists$ finite $S \subseteq A$, $S = \{s_1, \cdots s_k\}$ s.t. $A = \text{Span}_B \{ s_1^{a_1} s_2^{a_2} \cdots s_k^{a_k} \mid a_i \geq 0 \}$

A is finite over B:

$\exists$ finite $S \subseteq A$ $S = \{s_1, \cdots s_k\}$ s.t. $A = \text{Span}_B \{ s_1, s_2, \cdots s_k \}$.

Examples: 1) If L/K is f.d., L is a finite K-algebra.

2) K a field. Consider the K-algebra $A = K[T, T^{-1}]$. It's also an alg over $K[T]$, $K[T^{-1}]$, $K[T^{-1}, T]$

a) $K[T, T^{-1}]$ is not finite as a K-algebra.

let $S \subseteq K[T, T^{-1}]$ finite, then $S \subseteq \text{Span}_K \{ T^{-l}, T^{-l+1}, \cdots T^{l} \}$ for $l \in \mathbb{Z}$. not all of A.

b) $K[T, T^{-1}]$ is not finite as $K[T]$ algebra.

If $S \subseteq K[T, T^{-1}]$ is finite, $\text{Span}_{K[T]} S$'s exponents are bounded below.

But $K[T,T^{-1}]$ is finite as a $K[T-T^{-1}]$ algebra.

Indeed $\text{Span}_{K[T-T^{-1}]}(1,T) = A$.

$T^2 = (T-T^{-1})T + 1$

$T^{-1} = -(T-T^{-1}) + T$.

generally $T^{m+1} = (T-T^{-1})T^m + T^{m-1}$

$T^{m-1} = -(T-T^{-1})T^m + T^{m+1}$

<u>defn. let A be a B-alg. Then $a \in A$ is integral over</u> B if $f(a)=0$ for monic f in B.

A is integral over B if all $a \in A$ is integral over B.

note: for K-algebra A, x is algebraic over K $\Leftrightarrow$ integral over K.

algebraic & integral

↳ field ↳ ring

↳ any poly ↳ monic poly.

<u>lem. let C be an $n \times n$ matrix over a ring A.</u> let v be a col vec $v \in A^n$ s.t. $Cv = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ then $\det(C) \cdot v = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

mult by adj mat $\text{adj}(C) \cdot Cv = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(C) \cdot I \cdot v = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(C)\, v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

<u>Prop let A be an B algebra Then TFAE</u>

i) A is finitely generated integral B algebra.

ii) A is generated as a B-algebra by a finite set of B-integral elements.

iii) A is finite B algebra.

Proof: i) $\Rightarrow$ ii) A is finitely generated by $S = \{s_0, s_1, \cdots s_m\}$ each s_i is integral over B. So ii is true.

ii) ⇒ iii) Idea: higher powers of a_i can be written as lin comb of smaller powers of A.

let $\{a_1, \cdots a_s\}$ generate A as a B-alg. each a_i integral so

$a_i^{n_i} + b_{i-1} a^{n_i-1} + \cdots + b_0 = 0$ So $a_i^{n_i} \in \mathrm{span}\{a^{n_i-1}, \cdots a^0\}$.

$A = \mathrm{span}_B \{a_1^{e_1} \cdots a_s^{e_s}, e_i \geq 0\}$

so $A = \mathrm{Span}_B \{a_1^{x_1} \cdots a_s^{x_s} \quad 0 \leq x_i < n_i\}$ ← finite set.

iii) → i)

A is a finite B-alg ⇒ A is f.g., integral B-alg.

we know finite implies finitely generated. It suffice to show integral. let $\alpha \in A$. write $\varphi: B \to A$ as the structural hom, Consider $\varphi(B)[\alpha]$. Note, Since A is finite as a B-alg, A is a finite $\varphi(B)[\alpha]$ module.

(A is a faithful $\varphi(B)[\alpha]$ module b/c if $x \in \varphi(B)[\alpha]$, $xy=0 \ \forall y \in A$, then $y=1 \in A$, $x=0$)

using lemma, α is $\varphi(B)$ integral. So ∃ monic poly, f in $\varphi(B)$ s.t. $f(\alpha)=0$. But i.e. $f = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$.

$b_i \in \varphi(B)$

$\boxed{\text{why imply B- algebra ???}}$

lem. let $B \subset A$ be rings. Then $x \in A$ is integral over B iff ∃ a $B[x]$ submodule M of A s.t.

i) M is a faithful $B[x]$-module

ii) M is f.g. as a B-module.

Pf)

⇐ assume i), ii) holds.

then M is a B-module generated by $\{e_1, \cdots, e_n\}$

$xM \subset M$. M is faithful as $B[x]$ module.

but $x\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = C\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ for some $C \in B^{n \times n}$

$(xI - C)\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = 0$ So $\det(xI-C)\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = 0$ by prev lemma.

so $\det(xI-C) \cdot m = 0 \ \forall m \in \mathrm{span}_B\{e_1, \cdots e_n\}$.

$= 0$ b/c it's a faithful mod.

⇒ x int over B,

$\Rightarrow$ Say x integral over B.

say $x^n + b_1 x^{n-1} + \cdots + b_n x^0 = 0$

consider $M = \mathrm{Span}_B \{x^0, \dots, x^{n-1}\}$ is a B-submodule, $xM \subset M$.

So M is a $B[x]$-submodule of A. it's f.g. it's also faithful as $1 \in M$.

Proof schemes of lemmas and thm

Theorem: i) $\Rightarrow$ ii) f.g. integral

$\Rightarrow$ f.g. by finite B-integral elements.

ii) $\Rightarrow$ iii) say B is f.g. by finite B-int ele $\{s_0, \dots s_n\}$.

then each s_n writes as poly in B. Take that power to be n_i.

then A is f.g. as a $\{s_0^{a_0} s_1^{a_1} \cdots s_n^{a_n} \mid 0 \le a_i < n_i\}$-module

iii) $\Rightarrow$ i) use lemma f.g. easy. but integral needs lemma.

show A is finite $\varphi(B)[\alpha]$ module.

lemma: integral $\Leftrightarrow$ $B[x]$ module of A s.t. 1) faithful 2) finite as B-module.

$\Rightarrow$ say $a_n x^n + \cdots + a_0 = 0$ take $M = \mathrm{Span}_B \{x^0, \dots x^{n-1}\}$

$\Leftarrow$ $xM \subset M$, faithful, $x \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = C \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ use det trick to cook up the monic poly.

def. Algebraic independence

let A be an algebra over field K.

then $x_1, \dots x_n$ are alg. indep if the only poly $p \in K[T_1, \dots T_n]$ s.t.

$p(x_1, \dots x_n) = 0$ is 0.

$\Leftrightarrow$ $K[T_1, \dots T_n] \to A$ is injective

$\quad t_i \mapsto x_i$

In this case, the K-subalg of A generated by $x_1, \dots x_n \cong A[T_1, \dots T_n]$.

Thm Noether's Normalization theorem

we induct on the minimal # of generators of A as a K-algebra.

Base case: 0 generators. If $A = K$, $n=0$, $A = A'$.

Inductive hypothesis:

Assume that $x_1, \cdots x_m$ generate A as a K-algebra & theorem holds when A is generated as a K-alg with $< m$ generators.

If $x_1, \cdots x_m$ are algebraically independent, we're done.

Now, say $x_1, \cdots x_m$ are not algebraically independent.

Claim (to be proven later):

there exists $c_1, \cdots c_{m-1} \in K$ s.t. x_m is integral over $B = K[x_1 - c_1 x_m, x_2 - c_2 x_m, \cdots x_{m-1} - c_{m-1} x_m]$

then $A = B[x_m]$ By prop, A is f.g. B-alg, x_m int over B, $\Rightarrow$ A is finite over B.

But B is finite over $A' = K[y_1, \cdots y_n]$ for some n, so A is finite over A'.

Kinda like: if alg indep, good. n dimensions

if not alg indep, reduce case to $m-1$ variables, and less dims.

now: proof of claim: if $x_1, \cdots x_m$ are not alg. indep, then, $\exists\, c_1, \cdots c_{m-1}$ s.t.

x_m is integral over $B = [x_1 - c_1 x_m, x_2 - c_2 x_m, \cdots, x_{m-1} - c_{m-1} x_m]$

take $0 \neq f \in K[T_1, \cdots T_m]$ s.t. $f(x_1, \cdots x_m) = 0$.

write f as sum of its homogeneous parts.

let F be the part of highest degree. let r be its highest deg.

now, for $c_1, \cdots c_{m-1} \in K$ (pick later), we "bump up the degree" by this:

denote $g(T_1, \cdots, T_m)$: write as a T_m polynomial,

$$g(T_1 \cdots T_m) = f(T_1 + c_1 T_m, T_2 + c_2 T_m, \cdots, T_{m-1} + c_{m-1} T_m, T_m) = \underbrace{F(c_1, \cdots c_{m-1}, 1)}_{\text{"collect" the coeff terms}} T_m^r + \{\text{terms of deg} < r \text{ in } T_m\}.$$

note $g \in K[T_1, \cdots T_m]$, $g(x_1 - c_1 x_m, \cdots, x_{m-1} - c_{m-1} x_m, x_m) = f(x_1, \cdots, x_m) = 0$ | poly of T_m w/ coeff $\in K[T_1, \cdots T_{m-1}]$

consider g as a polynomial of T_m in $k[T_1, \cdots T_{m-1}]$, the leading coefficient of T_m^r is scalar $F(c_1, \cdots, c_{m-1}, 1)$. It's nonzero b/c $F(T_1, \cdots T_m)$ is a nonzero hom polynomial. i.e. all terms degree r. so $\exists c_1, \cdots c_{m-1}$ s.t. $F(c_1, \cdots c_{m-1}, 1) \neq 0$.

Proof scheme.

- induct on minimal generators m
- for $m=0$, ✓

  for $m>0$, say A is f.g. by $\{x_1, \cdots x_m\}$
  - alg indep. ✓
  - alg dependent. say $f \in k[T_1, \cdots T_m]$ $f(x_1, \cdots x_m) = 0$.

    claim, $\exists$ $c_1, \cdots c_{m-1}$ s.t. x_m is integral over $B = k[c_1x_1 - c_m, c_2x_2 - c_m, \cdots, c_{m-1}x_{m-1} - c_m]$

    so $A = B[x_m]$. But prop says f.g. & integral $\Rightarrow$ finite.

    use inductive hyp on B. done.
  - to show claim,

    $f(x_1, \cdots x_m) = 0$. write f in hom. components. Say $F(x_1, \cdots x_m)$ is the highest component of degree r.

    now, write

$$g(x_1, \cdots x_m) = f(x_1 + c_1 x_m, \cdots, x_{m-1} + c_{m-1} x_m, x_m)$$

$$= \underbrace{F(c_1, \cdots c_{m-1}, 1)}_{\substack{\text{nonzero b/c} \\ \text{of homogenous.}}} x_m^r + \underbrace{\qquad}_{< \deg r \text{ in } x_m.}$$

$$\underbrace{g(x_1 - c_1 x_m, \cdots, x_{m-1} - c_{m-1} x_m, x_m) = 0.}$$ So x_m integral over these.

Proof scheme for claim

- $f(x_1, \cdots x_m) = 0$, write into hom parts extract highest degree.
- construct g so claimed $x_i - c_i x_m$.
- g is a poly in x_m, but also $g(\_\_ - - -) = 0$

But ↑ infinite field. Try to prove for finite field.

lem: $p \in k[T_1, \cdots T_n]$. nonzero, degree d. k field. $S \subset k$ finite. Then,

$$|\{(x_1, \cdots x_n) \in S^n \mid f(x_1, \cdots x_n) = 0\}| \leq d|S|^{n-1}$$

pf : $n=1$, $(x_1) \in S$ $f(x_1)=0$, at most d sdns ✓

$n>1$.

write $P(x_1, \cdots x_n)$. If x_1 take value S_1, then rest have at most $d|S|^{n-2}$ values.

so on & so at most $d|S|^{n-1}$ values.

Section 4. Hilbert's NSZ.

Motivation: there's a bijection between k^n and $\hom_{k\text{-alg}}(k[T_1, \cdots T_n], k)$

$$k^n \rightleftharpoons \hom_{k\text{-alg}}(k[T_1, \cdots T_n], k)$$

$\longrightarrow$

given $(x_1, \cdots x_n) \in k^n$ define the k-alg hom $f: k[T_1 \cdots T_n] \to k$, $T_i \mapsto x_i$

$\longleftarrow$

gives a k-alg homomorphism, $f: k[T_1, \cdots T_n] \to k$. define $x_i = f(T_i)$

Idea: the bijection between k^n and $\mathrm{Hom}_{k\text{-alg}}(k[T_1, \cdots T_n], k)$

$k^n \rightleftharpoons$ k-algebra homomorphisms $(k[T_1, \cdots T_n], k)$

$x \to f_x$

$x_f \leftarrow f$

$f_{x_f} = f$ gives f, $x_f = [f(T_1), \cdots f(T_n)]$

then f_{x_f} is unique hom $T_1 \to f(T_1)$, $T_2 \to f(T_2)$ ⋮

$x_{f_x} = x$ gives $(x_1, \cdots x_n)$

f_x is sending $T_1 \to x_1, \cdots T_n \to x_n$

get it back.

Prop $\ker f_x = (T_1 - x_1, \dots, T_n - x_n)$

gives k-alg hom $f: k[T_1, \dots, T_n] \to k$,

you get a map $\mathrm{Hom}_{k\text{-alg}}(k[T_1, \cdots T_n], k) \to \mathrm{Id}(k[T_1 \cdots T_n])$

$f \mapsto \ker(f)$

want to show $\ker f_x = (T_1 - x_1, \cdots, T_n - x_n)$

$\supseteq$: if $g \in (T_1 - x_1, \cdots, T_n - x_n)$

then $f_x(g)$ = substitute T_i with x_i, no constant terms $\to$ so $f_x(g) = 0$

$g \in \ker f_x$.

$\subseteq$: if $p \in \ker f_x$.

f_x: gives a poly, replace T_i w/ x_i

p is a polynomial s.t. $f_x(g) = 0$, $p(x_1, \dots, x_n) = 0$

define $q(T_1, \cdots, T_n) = p(T_1 + x_1, T_2 + x_2, \cdots T_n + x_n)$

$q(T_1 - x_1, \cdots, T_n - x_n) = p(T_1, \cdots, T_n)$

$q(0, 0, \dots 0) = q(x_1 - x_1, \cdots, x_n - x_n) = p(x_1, \cdots x_n) = 0$

$\Rightarrow q$'s constant term is 0.

$p(T_1, \cdots T_n) = q(T_1 - x_1, \cdots T_n - x_n)$ constant term is 0, so $p \in (T_1 - x_1, \cdots, T_n - x_n)$.

Prop $(T_1 - x_1, \cdots T_n - x_n)$ is a maximal ideal.

note: $f_{\bar{x}}: k[T_1, \cdots T_n] \to k$

so $k[T_1, \cdots T_n] / \ker f_{\bar{x}} \cong \mathrm{im}(k) \underset{\uparrow}{=} k$

$f_{\bar{x}}$ is surjective

k is a field so $\ker f_{\bar{x}}$ is a max. ideal.

$\ker f_{\bar{x}} = (T_1 - x_1, \cdots, T_n - x_n)$

Prop: $(x_1, \cdots, x_n) \mapsto (T_1 - x_1, \cdots, T_n - x_n)$ is injective

$k^n \to \mathrm{mspec}(k[T_1, \cdots T_n])$

$(x_1, \cdots x_n) \mapsto (T_1 - x_1, \cdots, T_n - x_n)$

b/c $(x_1, \dots, x_n)$ is unique point in $V((T_1 - x_1, \cdots, T_n - x_n))$

but not surjective in general. (if k is alg closed, then it's surjective).

when K is alg closed,

$$K^n \to \text{mspec}(K[T_1, \dots T_n])$$

$(x_1, \dots, x_n) \mapsto (T_1 - x_1, \dots T_n - x_n)$ not surjective.

note: $(T^2+1) \subseteq \mathbb{R}$ is a max ideal.

but (T^2+1) is not of form $(T-x)$

idea: map $K^n \to \text{mspec}(K[T_1, \dots, T_n])$

$(x_1, \dots x_n) \mapsto (T_1 - x_1, \dots, T_n - x_n)$ is inj.

If not alg closed, give an ex s.t. not surjective

left off at 4.1.2. (Dec 14).

Dec 20. Pickup @ 4.1.2.

intuition for strong Nullstellensatz: $I(V(T^2)) = (T) \Rightarrow I(V(\cdot))$ is like taking roots.

def radical of an ideal

let a be an ideal of R then

$$\sqrt{a} = \{r \in R \mid r^n \in a, n \geq 0\}.$$

Strong nsz: $I(V(a)) = \sqrt{a}$.

Strong nsz: map V: {radical ideals of $L[T_1, \dots T_n]$} $\to$ {alg subset of L^n}

injective. (nsz)

surjective (by standard algebra)

so nsz: $I(V(a)) = a$ for all radical ideal a.

$V(I(X)) = X$ for all alg set X. by set theory.

prop: integral domain has cancellation property

Prop. let $A \subset B$ be integral domains. B integral over A. Then $A \cap B^\times = A^\times$

$A^\times \subseteq A \cap B^\times$: If $a \in A^\times$ then $a \in A$. $\exists a^{-1} \in A$, $aa^{-1} = 1$, so $a, a^{-1} \in B$ so $a \in B^\times$.

$A \cap B^\times \subseteq A^\times$: let $a \in A \cap B^\times$

since $a \in B^\times$ $\exists b \in B$, $ab = 1$ NTS $b \in A$.

indeed, $\exists a_1, \cdots a_n \in A$, $b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$

$\times a^{n-1}$ $b + \underbrace{a_{n-1} + a_{n-2} \cdot a + \cdots + a_0 \cdot a^{n-1}}_{\in A} = 0$

lem $A \subset B$ integral domains. B integral over A. Then B is field $\Leftrightarrow$ A is a field.

$\Rightarrow$ B is a field. Then $A^\times = A \cap B^\times = A \cap (B \setminus \{0\}) = A \setminus \{0\}$ so A is a field.

$\Leftarrow$ say A is a field.

let $0 \neq b \in B$ be arbitrary. 0

note $\exists a_0, \cdots a_{n-1} \in A$, n minimal, $n \geq 0$ s.t.

$$b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$$

$$b \underbrace{(b^{n-1} + a_{n-1} b^{n-2} + \cdots + a_1)}_{\Delta} = -a_0$$

$\Delta \neq 0$ by minimality of n. and $(-a_0)$ has inverse so

$b \left(\Delta \cdot (-a_0)^{-1} \right) = 1$ so b has inverse.

Prop Zariski's lemma.

let $k \subset K$ be fields. K is finitely generated as a k-algebra.

then K is finite as a k-algebra ($\dim_k K < \infty$)

Proof.

use Noether's normalization thm on K. so K is finite over $A = k[x_1, \cdots x_n]$ where $x_1, \cdots x_n$ are algebraically indep. K is integral over A by prev. lemma, A is also a field. so $n=0$, so K is finite over k.

Thm Weak Nullstellensatz

o.w. contains 1.

for a field k, a proper ideal $\mathfrak{a}$ of $k[T_1, \cdots T_n]$, there's a field extension L of k, $x \in L^n$ s.t. $f(x)=0$ $\forall f \in \mathfrak{a}$. (if k is alg-closed, $L=k$)

Pf. let m be a max ideal of $k[T_1, \cdots T_n]$ that contains $\mathfrak{a}$.

then $L = A/m$ is a field, and a k-algebra generated by $T_1+m, T_2+m, \cdots T_n+m$.

consider the point $x = (T_1+m, \cdots\cdots, T_n+m) \in L^n$

then, $x \in V(\mathfrak{a})$ b/c $\forall f \in \mathfrak{a}$, $f(x) = f(T_1+m, \cdots T_n+m)$

$$= f(T_1, \cdots T_n) + (m) = 0+m$$

$\in \mathfrak{a} \subseteq m$

$\dim_k L < \infty$ by Zariski's lemma

look at this direction later?

note: pg 13 remark: see later.

effective nullstellensatz: the only way blocking us from finding a solution is $\exists\, r_i$ s.t. $r_i p_i = 1$?

Cor algebraically closed implies bijection between points & mspec.

If k is alg closed field.

Then $k^n \to \operatorname{mspec}(k[T_1, \cdots T_n])$

$(x_1, \cdots x_n) \mapsto (T_1 - x_1, \cdots T_n - x_n)$ is a bijection.

Why is $(T_1-x_1, \cdots T_n-x_n) \in \operatorname{mspec}$?

consider $k^n \rightleftarrows \hom_k(k[T_1, \cdots T_n], k)$.

this is a bijection.

$f_{\bar{x}} : k[T_1, \cdots T_n] \to k$

$k[T_1, \cdots T_n] / \underbrace{\ker f_{\bar{x}}}_{\text{max ideal}} \cong \operatorname{Im} f_{\bar{x}} = \underset{\text{field.}}{k}$

injectivity: $(x_1, \cdots x_n)$ is unique point in $V((T_1-x_1, \cdots T_n-x_n))$

surjectivity: $m \in \operatorname{mspec}(k[T_1, \cdots T_n])$

By 4.3, k is alg closed, $\exists\, x \in k^n$ s.t. $x \in V(m)$.

let $m_x = (T_1 - x_1, \cdots T_n - x_n)$

claim $m_x = m$

$m_x \subset m$: b/c m is maximal and m_x is proper ideal of m

$m \subset m_x$: let $f \in m$. then $f(x) = 0$.

let $g(T_1, \cdots T_n) = f(T_1 + x_1, \cdots T_n + x_n)$

$g(0, \cdots 0) = 0$ $\Rightarrow$ g's constant term is 0.

$\Rightarrow$ $f(T_1, \cdots T_n) = g(T_1 - x_1, \cdots, T_n - x_n)$ is a poly in $T_i - x_i$ with 0 as constant term. $\Rightarrow$ $f \in (T_1 - x_1, \cdots T_n - x_n)$.

def. $V_{k^{al}}(a) = \{x \in (k^{al})^n \mid f(x) = 0 \ \forall f \in a\}$.

dimension

thm Strong Nullstellensatz.

$$I(V_{k^{al}}(a)) = \sqrt{a}$$

$\supseteq$: $I(V_{k^{al}}(a))$ is radical ($I(X)$, X any alg subset is radical)

and $a \subset I(V_{k^{al}}(a))$ so $\sqrt{a} \subseteq I(V_{k^{al}}(a))$

$\subseteq$:

Statement of Strong NSZ:

let a be an ideal of $k[T_1, \cdots T_n]$. k a field.

let $h \in k[T_1, \cdots T_n]$ s.t. $h(x) = 0$ $\forall x \in V_{k^{al}}(a)$. Then $\exists$ $l \geq 1$ s.t. $h^l \in a$.

Proof if $h = 0$ clear

assume $h \neq 0$. By Hilbert's basis thm, $\exists g_1, \cdots g_m \in k[T_1, \cdots T_n]$ s.t. $a = (g_1, \cdots g_m)$

then consider $b = (g_1, \cdots g_m, 1 - Yh)$ of $k[T_1, \cdots, T_n, Y]$.

Claim $V_{k^{al}}(b) = \phi$. Proof: if $(x_1, \cdots x_n, y)$ have $g_i(x_1, \cdots x_n) = 0$, then this point in $V(a)$) so that $h(x_1, \cdots x_n) = 0$ then Yh must evaluate to 0 on this point. so it's got 1.

since there's no point in $V_{k^{al}}$ that lie in $V(b)$, weak NSZ $\Rightarrow$ b is not a proper ideal, so $1 \in b$.

$\Rightarrow$ "polynomial liner comb" $\exists$ $r_1, \cdots r_m \in k[T_1, \cdots, T_n, Y]$ s.t.

$$1 = \left(\sum_{i=1}^{m} r_i g_i\right) + (r_{m+1}) \cdot (1 - Yh)$$

r_i are poly in $(T_1, \cdots T_n, Y)$ But h, g_i are in $k[T_1, \cdots T_n]$.

$\hookrightarrow$ ring homomorphism

$$k[T_1, \cdots, T_n, Y] \to k(T_1, \cdots, T_n)$$
$$T_i \mapsto T_i$$
$$Y \mapsto h^{-1} \quad \text{(well defined, } h \neq 0\text{)}.$$

apply the ing homomorphism to $\sum_{i=1}^{m} r_i g_i + r_{m+1}(1-Yh)$

get $1 = \sum_{i=1}^{m} r_i(T_1, \cdots, T_n, h^{-1}) g_i$

for large enough l, $h^l r_i(T_1, \cdots T_n, h^{-1}) \in K[T_1 \cdots, T_n]$.

$1 = \sum_{i=1}^{m} r_i(T_1, \cdots, T_n, h^{-1}) g_i$

multiply both side h^l

$$\Rightarrow h^l = \sum_{i=1}^{m} \underbrace{h^l r_i(T_1, \cdots, T_n, h^{-1})}_{k[T_1, \cdots, T_n]} \underbrace{g_i}_{\in a}$$

$\Rightarrow h \in \sqrt{a}$.

Proof scheme:

claim: if $h \in I(V_{k^{al}}(a))$ i.e. $h(x) = 0$ $\forall x \in V_{k^{al}}(a)$ then $h \in \sqrt{a}$.

Proof: let $h(x) = 0$ $\forall x \in V_{k^{al}}(a)$.

↳ hilbert Basis: let $a = (g_1, \cdots g_m)$ as ideal, so $h \in (g_1, \cdots g_m)$

↳ introduce $b = (g_1, \cdots, g_m, 1 - Yh) \in k[T_1, \cdots, T_n, Y]$.

↳ nothing in $V_{k^{al}}(b)$ $\Rightarrow$ weak NSZ $\Rightarrow$ $1 \in$ ideal

↳ $\exists$ "lin comb" to 1.

↳ apply hom: $k[T_1, \cdots T_m, Y] \to R(T_1, \cdots, T_m)$

↳ bump up degree $\Rightarrow$ $h^l \in a$.

Strong NSZ:

↳ $I(V_{k^{al}}(a)) \subset \sqrt{a}$

Easier claims:

↳ $I(V_{k^{al}}(a)) \supseteq \sqrt{a}$

↳ $V(I(X)) = X$

So we get the bijection

$\{$radical ideals of $k[T_1, \cdots, T_n]\} \leftrightarrow \{$alg subsets of $k^n\}$

$$\mathfrak{a} = \sqrt{\mathfrak{a}} \longmapsto V(\mathfrak{a})$$

$$I(X) \longmapsfrom X$$

This bijection is order-reversing: $X \subset Y \Rightarrow I(X) \supset I(Y)$

$\mathfrak{a} \subset \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supset V(\mathfrak{b})$

hicup: skipped excercise #1.

Chapter 5. The Zariski topology on k^n & spec(A).

Prop. Let k^n be a field. Then, the topology where closed sets are defined by algebraic subsets form defines a topology on k^n. (Zariski topology).

Pf: suffices to show open sets ($\phi, X \in$ top, finite $\cap$, infinite $\cup$)

$\Leftrightarrow$ closed set $\phi, X \in$ top, arbitrary $\cap$, finite $\cup$).

Proof: 1) k^n, ϕ are closed b/c $k^n = V((0))$ $\phi = V((1))$

2) let $X = V(\mathfrak{a})$, $Y = V(\mathfrak{b})$ in top, then $X \cup Y = V(\mathfrak{a}) \cup V(\mathfrak{b})$

$= V(\mathfrak{a}\mathfrak{b})$ ideal products, not necessarily literally the product.

3) let $(\mathfrak{a}_i)_{i \in I}$ be ideals, $\bigcap_{i \in I} V(\mathfrak{a}_i) = V(\sum_{i \in I} \mathfrak{a}_i)$

$\subseteq$: let $x \in \bigcap_i V(\mathfrak{a}_i)$ then any sum of elements among diff $\mathfrak{a}_i$ will kill x.

$\supseteq$: let $x \in V(\sum_{i \in I} \mathfrak{a}_i)$ then $x \in \mathfrak{a}_i$ (set 0 to $j \neq i$,

Rmk $D(f) = \{x \in k^n, f(x) \neq 0\}$ is open.

$D(f)$ is basis of open sets of k^n.

def Hausdorff : every two points is separated by two open sets.

Rmk: Hausdorfness of the space.

K is finite $\Rightarrow \mathbb{A}_k^n$ is Hausdorff.

K is infinite $\Rightarrow \mathbb{A}_k^n$ is not Hausdorff: $\phi \neq \{x \in k^n \mid f(x) \neq 0\}$.

since $D(f)$ defines basis, but $\phi \neq D(f) \subset U$, $\phi \neq D(g) \subset V$, and $\phi \neq D(fg) \subset U \cap V$

$fg(x) \neq 0$ so $f(x) \neq 0 \neq g(x)$

$\Rightarrow x \in U \cap V$.

---

def irreducible top. space.

A top space is irred if $\nexists$ closed proper subsets X_1, X_2 s.t. $X = X_1 \cup X_2$.

Prop irreducible $\Leftrightarrow$ every two nonempty subsets of X intersect.

Proof $\nexists$ closed sets X_1, X_2, $X = X_1 \cup X_2$. $\Rightarrow$ opensets intersect.

contrapositive U_1, U_2 open. $U_1 \cap U_2 = \phi$

$(U_1 \cap U_2)^c = X \Rightarrow U_1^c \cup U_2^c = X$

opens intersect $\Rightarrow \nexists$ closed sets X_1, X_2, $X = X_1 \cup X_2$.

contrapositive: X_1, X_2 closed proper, $X = X_1 \cup X_2$. $\phi = (X_1 \cup X_2)^c = X_1^c \cap X_2^c$

Remarks $\mathbb{A}_n^k$, $\mathrm{char}(k)=0$ is not Hausdorff. $\Rightarrow$ irreducible

1) every singleton of K^n is irreducible

2) Hausdorff irreducible $\Leftrightarrow$ singleton.

$\Leftarrow \checkmark$ $\Rightarrow$ not singleton. $X_1, X_2 \in X$. ?

3) let $p, q \in K[T_1, \cdots T_n]$ be irreducible, $p \neq cq$ $\forall c \in K$. then $X = V(p)) \cup V((q))$ is not irreducible.

note $V((p)) \not\subseteq V((q))$ & $V((q)) \not\subseteq V((p))$ $\quad p = \sqrt{p}$, $q = \sqrt{q}$

(otherwise $V((p)) \subseteq V((q)) \Rightarrow I(V(p)) \supseteq I(V((q))$ $\not\downarrow$.

So $X = V((p)) \cup V((q))$ is not irreducible

Recall: $K[T_1, \cdots T_n]$ is a UFD for K a field.

Exercise pg 17. not completed!

HICCUP ?

lemma let P be a prime ideal. let $I_1, I_2 \subset R$ be ideals. if $I_1 \cap I_2 \subset P$ then either $I_1 \subset P$ or $I_2 \subset P$.

Proof: Say $I_1 \not\subset P$, $I_2 \not\subset P$ then $\exists f \in I_1 \setminus P$, $g \in I_2 \setminus P$ then $fg \in I_1 \cap I_2 \subset P$ ※.

Prop: let k be a field. An alg set $X \subseteq \mathbb{A}_k^n$ is irreducible $\Leftrightarrow$ $I(X)$ is prime.

$\Rightarrow$ let X be irreducible. let $fg \in I(X)$ $\quad X \subseteq V(f) \cup V(g)$. Then, $X = (X \cap V(f)) \cup (X \cap V(g))$

X is irred, so either $X = X \cap V(f)$ or $X = X \cap V(g)$. Say $X = X \cap V(g)$ $\quad X \subseteq V(g)$. $g \in I(X)$

$\Leftarrow$ let $I(X)$ be a prime ideal. WTS X is irred. let $X = \underbrace{X_1 \cup X_2}_{\text{closed}}$ $\quad I(X) = I(X_1) \cap I(X_2)$ But $I(X)$ prime so either $I(X_1) \subset I(X)$ or $I(X_2) \subset I(X)$.

Say $I(X_1) \subset I(X)$. But $X_1 \subset X$. $I(X_1) \supset I(X)$ $\Rightarrow I(X) = I(X_1)$ $\quad \underset{=X}{V(I(X))} = \underset{=X_1}{V(I(X_1))}$

strong NSZ $\Rightarrow$ alg closed $\Rightarrow$ $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ $\quad V(\mathfrak{a})$ is irred $\Leftrightarrow$ $\sqrt{\mathfrak{a}}$ prime.

Remark: counter ex for ↑ when k is not alg closed.

Chapter 6 the space spec(A)

def. the Zariski topology on spec (A) $\quad A$ is a ring

closed sets: $V(\mathfrak{a}) = \{ p \in \operatorname{spec}(A) \mid \mathfrak{a} \subset p \}$

open sets: $D(f) = \{ p \in \operatorname{spec}(A) \mid f \notin p \}$.

→ HICCUP ☆ see example sheets to prove this is a topology. !!! ☆☆ ☆.

Chapter 7. localisation

def. multiplicative subset

In a ring R, a multiplicative subset $S \subseteq R$ is a set, s.t. $1 \in S$, and $\forall a, b \in S$, $ab \in S$.

def. $\sim$ on $A \times S$

$(a_1, s_1) \sim (a_2, s_2)$ if $t(a_1 s_2 - a_2 s_1) = 0$ for some $t \in S$.

let $S^{-1}A$ define $A \times S / \sim$.

Prop $\times$ and $+$ are well-defined on $S^{-1}A$.

1) $+$ is well-defined:

$\frac{a_1}{s_1} = \frac{b_1}{t_1}$ $\quad \frac{a_2}{s_2} = \frac{b_2}{t_2}$ in $S^{-1}A$.

so $\exists u, v$ s.t. $u(a_1t_1 - b_1s_1) = 0$ $\quad v(a_2t_2 - b_2s_2) = 0$

$$\frac{a_1s_2 + s_1a_2}{s_1s_2} \overset{?}{=} \frac{b_1t_2 + t_1b_2}{t_1t_2}$$

$$t_1t_2(a_1s_2 + s_1a_2) - s_1s_2(b_1t_2 + t_1b_2)$$

$$= s_2t_2(a_1t_1 - s_1b_1) + s_1t_1(s_2t_2 - s_2b_2)$$

$\times uv = 0.$

2) $\times$ is well defined

$\frac{a_1}{s_1} = \frac{b_1}{t_1}$ $\quad \frac{a_2}{s_2} = \frac{b_2}{t_2}$

so $\exists u, v$ s.t. $u(a_1t_1 - b_1s_1) = 0$ $\quad v(a_2t_2 - b_2s_2) = 0$

$$\frac{a_1a_2}{s_1s_2} \overset{?}{=} \frac{b_1b_2}{t_1t_2}$$

$$a_1a_2t_1t_2 - s_1s_2b_1b_2 \overset{?}{=} 0$$

$$a_1t_1a_2t_2 - b_1s_1a_2t_2 + b_1s_1a_2t_2 - s_1s_2b_1b_2$$

$$= (a_2t_2)(a_1t_1 - b_1s_1) + b_1s_1(a_2t_2 - s_2b_2) = 0 \quad \checkmark$$

def $i_S : A \to S^{-1}A : a \mapsto \frac{a}{1}$

note i_S sends S to units.

Prop universal property of $S^{-1}A$

1) $\forall s \in S$, $i_S(s)$ is a unit.

2) let B be a ring and $f : A \to B$ be a ring hom s.t. $f(s)$ is a unit $\forall s \in S$, then there's a unique hom $h : S^{-1}A \to B$ s.t. $f = h \circ i_S$. & $h(\frac{a}{s}) = h(a)h(s)^{-1}$ $a \in A$, $s \in S$.

$$\begin{tikzcd} A \arrow[r, "i_S"] \arrow[rr, bend right, "f"'] & S^{-1}A \arrow[r, "h"] & B \end{tikzcd}$$

Proof: $i_S(s) = \frac{s}{1}$ but $\frac{1}{s} \in S^{-1}A$ so $\frac{s}{1} \cdot \frac{1}{s} = 1$

2) uniqueness: Suppose $\exists h: S^{-1}A \to B$ s.t. the diagram commutes, then $\forall$

$$f(a) = h(\tfrac{a}{1}) = h(\tfrac{a}{s})h(\tfrac{s}{1}) = h(\tfrac{a}{s})f(s)$$

$f(s)$ is a unit.

$$\underbrace{h(\tfrac{a}{s}) = f(a)\,f(s)^{-1}}$$

to show this is true:

existence: need to show $h(\frac{a}{s}) = f(a)f(s)^{-1}$ is well defined & defines a ring hom.

$\hookrightarrow$ well defined: $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ then $\exists\, u \in S$ s.t. $u(a_1 s_2 - s_1 a_2) = 0$.

so $f(u)\,[f(a_1)f(s_1) - f(s_1)f(a_2)] = 0$

$f(u)$ is a unit and $\neq 0$ so

$f(a_1)f(s_1) - f(s_1)f(a_2) = 0 \Rightarrow f(\frac{a_1}{s_1}) = f(\frac{a_2}{s_2})$

$\hookrightarrow$ $h(\frac{a}{s}) = f(a)f(s)^{-1}$ is a ring hom.

$\times$: $h(\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}) = h(\frac{a_1 a_2}{s_1 s_2}) = f(a_1 a_2) f(s_1 s_2)^{-1}$ ✓

$+$: $h(\frac{a_1}{s_1} + \frac{a_2}{s_2}) = h\left(\frac{a_1 s_2 + s_1 a_2}{s_1 s_2}\right) = f(a_1 s_2) f(s_1 s_2)^{-1} + f(s_1 a_2) f(s_1 s_2)^{-1}$

Proof scheme.

main msg: $f: A \to B$ s.t. f maps S to units then f factors thru uniquely

$$\begin{tikzcd} A \arrow[r, "i_S"] \arrow[rr, bend right, "f"'] & S^{-1}A \arrow[r, "h"] & B \end{tikzcd}$$

1. If such h exists, $h(\frac{a}{s}) = f(a)f(s)^{-1}$
2. h is well defined (choosing different representatives)
3. h is a homomorphism.

Rmk: (see example sheet).

If Q, $j: A \to Q$ has same universal property as $(S^{-1}A, i_S)$ then Q is isomorphic to $S^{-1}A$.

lem if $0 \in S$, then $S^{-1}A = \{0\}$.

bcus any $\frac{a}{b} = \frac{c}{d}$ i.e. $0(ad - bc) = 0$

def A_h as a ring

let $h \in A$. let $S_h = \{1, h, h^2, \cdots\}$ be multiplicative set.

then let $A_h = S_h^{-1} A$. if h is nil then $A_h = \{0\}$.

If A is ID, $h \neq 0$, then A_h is a subring of $\operatorname{Frac} A$.

w/ elements $\frac{a}{h^m}$, $m \in \mathbb{Z}$.

prop map $A[T]/_{1-hT} \longrightarrow A_h$:

let A be ring, $h \in A$, the map $(\sum_{i=0}^d a_i T^i) + (1-hT) \xrightarrow{\varphi} \sum_{i=0}^d \frac{a_i}{h^i}$ is well defined isomorphism.

Proof:

$$\varphi': A[T] \longrightarrow A_h$$
$$\sum_{i=0}^d a_i T^i \longrightarrow \sum_{i=0}^d \frac{a_i}{h^i}$$

note that $1-hT \in \ker \varphi'$. so $(1-hT) \subseteq \ker \varphi$.

so that $\varphi: A[T]/(1-hT) \longrightarrow A_h$ is well defined.

$$\psi': A \longrightarrow A[T]/(1-hT)$$

by $\psi'(a) = a + 1-hT$.

let $h^N \in S_h$. then $\psi'(h^N)$ has an inverse, $T^n + (1-hT)$.

i.e. $\psi'(h^N) = h^N + 1-hT$

$(h^N + (1-hT))(T^n + (1-hT)) = (hT)^N = (1-(1-hT))^N = 1$.

So, by universal prop, have
$$\begin{tikzcd} A_h \arrow[r, "i_{S_h}"] \arrow[rr, bend right, "\psi"] & A \arrow[r, "\psi'"] & A[T]/(1-hT) \end{tikzcd}$$

Two way homomorphism

Proof scheme:

Statement:

$$\boxed{A[T]/_{1-hT} \longrightarrow A_h}$$

$$\left(\sum_{i=0}^{d} a_i T^i\right) + 1-hT \longmapsto \sum_{i=0}^{d} \frac{a_i}{h^i}$$ is an iso

$$\varphi' : \boxed{A[T] \longrightarrow A_h}$$

$$\sum_{i=0}^{d} a_i T^i \longmapsto \sum_{i=0}^{d} \frac{a_i}{h^i}$$ with $1-hT \in \ker(\varphi')$

so $\varphi: A[T]/_{1-hT} \longrightarrow A_h$ is well defined

$$\boxed{\psi' : A_h \longrightarrow A[T]/_{1-hT}}$$

$$\boxed{\psi : A \longrightarrow A[T]/_{1-hT}}$$ with ψ sending all elements in S_h to units in the range. So by universal prop, ψ' factor thru to ψ.

<u>defn extension & contraction of ideals:</u>

$\varphi: A \to B$ hom.

<u>contraction of an ideal:</u>

for ideal b of B, $b^c = \varphi^{-1}(b)$ is contraction of b.

it's an ideal of A.

<u>extension of an ideal:</u>

let a be an ideal of A, then the ideal generated by $\varphi(a)$ is an ideal of b. denote by a^e.

<u>lem. let $A \subset B$. let $i = A \hookrightarrow B$ then $b^c = b \cap A$.</u>

$\varphi^{-1}(b) \subset b \cap A$ let $a \in \varphi^{-1}(b)$. $\Rightarrow a \in A$. $a \in b$.

$b \cap A \subset \varphi^{-1}(b)$ let $a \in b \cap A$. $\varphi(a) \in b$.

lem Surjective φ.

φ surjective, then $\varphi(a)$ is an ideal.

if φ surjective $\varphi: A \to B \trianglelefteq im(\varphi)$

$A/\ker\varphi \cong im\varphi = B.$

def. extended ideal & contracted ideal.

Prop $a \subset a^{ec}$, $b^{ce} \subset b$, $a^{ece} = a$, $b^{cec} = b$.

↳ let $x \in a$. WTS that $\varphi(x) \in (a^e)$ which is true.

↳ note that b^{ce} is an ideal generated by $\varphi(\varphi^{-1}(b))$.

Suffice to show that $\varphi(\varphi^{-1}(b)) \subset b$

let $x \in a$ s.t. $\varphi(x) \in b$ so $x \in \varphi^{-1}(b)$ $\varphi(x) \in b$ so $\varphi(\varphi^{-1}(b)) \subset b$.

↳ so $a^e \subset a^{ece}$ and $b^{cec} \subset b^c$

↳ $b^{ce} \subset b$, plug in $b = a^e$ get $a^{ecc} \subset a^e \Rightarrow a^e = a^{ece}$

↳ $a \subset a^{ec}$, plug in $a = b^c$ get $b^c \subset b^{cec} \Rightarrow b^c = b^{cec}$

So we get

$\{$ contracted ideals of $A \}$ $\longleftrightarrow$ $\{$ extended ideals of $B \}$

$b^c \mapsto b^{ce}$

$a^{ec} \mapsfrom a^e$

Remark: R/I is field $\Leftrightarrow$ I is maximal

ID prime

reduced radical

Prop let b be ideal of B. Then b is prime iff b^c is prime.

Consider $\varphi: A \to B/b$ note $\ker(\varphi) = b^c$ so

$A/b^c \cong B/b$. So b prime of B

$\Leftrightarrow$ $B/b \cong A/b^c$ is ID

$\Leftrightarrow$ b^c is prime

Rmk. Contracted / extended ideals in the context of localization.

let $i_S : A \to S^{-1}A$ be the hom.

let $\mathfrak{a}$ be an ideal of A, then $\mathfrak{a}^e = \{ \frac{x}{s} \mid x \in \mathfrak{a}, s \in S \}$

let $\mathfrak{b}$ be an ideal of $S^{-1}A$, then $\mathfrak{b}^c = \{ y \mid \frac{y}{1} \in \mathfrak{b} \}$.

Prop. let S, A be as above.

i) $\mathfrak{b}^{ce} = \mathfrak{b}$ $\forall$ ideal $\mathfrak{b}$ of $S^{-1}A$

ii) there's a bijection

$\{$ prime ideals of A that avoids $S \}$ $\longleftrightarrow$ $\{$ prime ideals of $S^{-1}A \}$.

$$\mathfrak{p} \longmapsto \mathfrak{p}^e$$
$$\mathfrak{q}^c \longleftarrow \mathfrak{q}$$

Proof.

i) $\mathfrak{b}^{ce} = \mathfrak{b}$

we know $\mathfrak{b}^{ce} \subset \mathfrak{b}$.

now need to show $\mathfrak{b} \subset \mathfrak{b}^{ce}$. let $\frac{a}{s} \in \mathfrak{b}$ so $a \notin \mathfrak{a}$ $s \in S$.

so $a \in \mathfrak{b}^c$ so $\frac{a}{1} \in \mathfrak{b}^{ce}$ $\frac{a}{s} \in \mathfrak{b}^{ce}$.

ii) need to show ① $\mathfrak{p}^{ec} = \mathfrak{p}$ and ② $\mathfrak{q}^{ce} = \mathfrak{q}$.

③ and $\mathfrak{p}^e$ are prime

④ and $\mathfrak{q}^c$ are prime that avoid S.

① $\mathfrak{p}^{ec} = \mathfrak{p}$ we know that $\mathfrak{p}^{ec} \supset \mathfrak{p}$.

need to show $\mathfrak{p}^{ec} \subset \mathfrak{p}$.

let $a_0 \in \mathfrak{p}^{ec}$ so $\exists$ $b_0 \in \mathfrak{p}^e$ s.t. $b_0 = \frac{a_0}{1}$ but $b_0 \in \mathfrak{p}^e$ so

$\frac{a_0}{1} = \frac{a}{s}$ for some $a \in \mathfrak{p}$, and $s \in S$. $\Rightarrow u(a_0 s - a) = 0$ $u \in S$

$\Rightarrow u a_0 s \in \mathfrak{p}$ but $u, s \notin \mathfrak{p}$ so $a_0 \in \mathfrak{p}$. so $\mathfrak{p}^{ec} \subset \mathfrak{p}$.

② $\mathfrak{q}^{ce} = \mathfrak{q}$ by i)

③ write $\bar{S}$ be S's image in $A/\mathfrak{p}$.

then $(S^{-1}A)/p^e \cong \bar{S}^{-1}(A/p)$ hiccup: check ex sheet.

$0 \notin \bar{S}$

since $S \cap p = \phi$, $0 \notin \bar{S}$. (o.w. if $x \in S \cap p$, then $x \in S$ is in image of S and map to 0 in S/p) $\Rightarrow 0 \notin \bar{S}$

so $0 \notin \bar{S}$ & A/p is ID $\Rightarrow (S^{-1}A)/p^e$ is ID $\Rightarrow p^e$ is prime.

④ q^c is prime ✓

q^c avoids S. If $s \in q^c$ then $i(s) \in q$ $\frac{s}{1} \in q$. $\frac{1}{s} \in S^{-1}A \Rightarrow 1 \in q$ ✗.

Proof scheme

↳ statement: bijection.

$\{$prime ideals of A avoiding $S\} \longleftrightarrow \{$prime ideals of $S^{-1}A\}$.

$$p \longmapsto p^e$$
$$q^c \longleftarrow q$$

① $p^{ec} = p$

② $q^{ce} = q$

③ p^e is prime. look at $(S^{-1}A)/p^e \cong \underbrace{\bar{S}^{-1}(A/p)}_{\text{ID.}}$

④ contradiction.

Prop Let p be a prime ideal of A. Write $S_p = A \setminus p$ a mult set.

$A_p = S_p^{-1}A$

↳ prime ideals of A disjoint from $S_p = A \setminus p$ are prime ideals contained in p.

↳ get bijection

$\{$prime ideals contained in $p\} \Leftrightarrow \{$prime ideals of $A_p\}$

note: $S_p^{-1}p$ of A_p is maximal ideal.

Example

$\mathbb{Z}_{(p)}$ is when $S = \mathbb{Z} \setminus (p)$ $S^{-1}A = \{ \frac{a}{b} \mid p \nmid b \}$.

prime ideal of $\mathbb{Z}$ contained in $\mathbb{Z}_{(p)}$ is (0) and $p\mathbb{Z}$.

$\Rightarrow \mathbb{Z}_{(p)}$ has two prime id: (0) and $p\mathbb{Z}_{(p)}$

$p\mathbb{Z}_{(p)}$ is max as for $x \in \mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$, it's invertible.

Chapter 8. going up & down.

Assume $A \subset B$ integral.

Prop $A \subset B$ integral. $\mathfrak{b}$ an ideal of B. then

$A/\mathfrak{b} \cap A \hookrightarrow B/\mathfrak{b}$ is an integral extension.

let $(x + \mathfrak{b}) \in B/\mathfrak{b}$. So $x \in B$.

$\exists\, a_0, \cdots a_n \in A$ s.t. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$

but $(x+\mathfrak{b})^n + (a_{n-1} + \mathfrak{b} \cap A)(x+\mathfrak{b})^{n-1} + \cdots + (a_0 + \mathfrak{b} \cap A) = 0 \mod \mathfrak{b}$.

Cor. let $A \subset B$ be integral extension. let $\mathfrak{q}$ be a prime ideal of B. then $\mathfrak{q} \cap A$ is a max ideal of $A \Leftrightarrow \mathfrak{q}$ is a max ideal of B.

Proof: using the cor "if $A \subset B$ are integral domains and integral extensions, then A field $\Leftrightarrow B$ field.

Consider map $A \hookrightarrow B \to B/\mathfrak{q}$. kernel of map is $A \cap \mathfrak{q}$.

So get $A/(A \cap \mathfrak{q}) \hookrightarrow B/\mathfrak{q}$.

$A \cap \mathfrak{q}$ is a prime ideal of A. (check) so $A/A\cap\mathfrak{q}$, $B/\mathfrak{q}$ are ID.

Also, $A/(A\cap\mathfrak{q}) \hookrightarrow B/\mathfrak{q}$ is integral extension

so we can use the thm. $A/(A\cap\mathfrak{q})$ is a field $\Leftrightarrow B/\mathfrak{q}$ is a field

So $A \cap \mathfrak{q}$ max $\Leftrightarrow \mathfrak{q}$ max.

Remark : relationship between $S^{-1}A$ and $S^{-1}B$.

let $A \subset B$, let S be a multiplicative set of A.

then $S^{-1}A$ is a subset of $S^{-1}B$.

then, $S^{-1}A \to S^{-1}B$ is a hom.

$A \to S^{-1}B$ $a \mapsto \frac{a}{1}$ sends all elements in S to units.

So $S^{-1}A \to S^{-1}B$ factors through.

kernel is $\{0\}$. ($\frac{a}{s} = 0$ in $S^{-1}A$ $\Leftrightarrow$ $\frac{a}{s} = 0$ in $S^{-1}B$).

$A \subset B$ integral $\Rightarrow$ $S^{-1}A \subset S^{-1}B$ integral

let $\frac{b}{s} \in S^{-1}b$, then $\exists$ $a_0, \cdots a_{n-1}$ s.t.

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$

$$\left(\frac{b}{s}\right)^n + \left(\frac{a_{n-1}}{s}\right)\left(\frac{b}{s}\right)^{n-1} + \cdots + \left(\frac{a_1}{s^{n-1}}\right)\left(\frac{b}{s}\right) + \left(\frac{a_0}{s^n}\right) = 0$$

Prop (Incomparability)

let $A \subset B$ be integral extension of rings. q, q' prime ideals of B s.t. $q \cap A = q' \cap A$ and $q \subseteq q'$. then $q = q'$.

Proof: let $p = q \cap A = q' \cap A$. let $S = A \setminus p$.

write $A_p = S^{-1}A$, $B_p = S^{-1}B$. Then $A_p \subset B_p$, B_p integral over A_p.

write $\begin{cases} pA_p \text{ for extension of } p \text{ to } A_p \\ qB_p \text{ '' '' } q \text{ '' } B_p \\ q'B_p \text{ '' '' } q' \text{ '' } B_p. \end{cases}$

↳ claim: $qB_p \subseteq q'B_p$ are prime ideals of B_p

because q, q' avoids $S = A \setminus p$.

↳ claim: $qB_p \cap A_p = pA_p$

$\supseteq$: $qB_p = \{ \frac{b}{s} \quad b \in q, \; s \in S \}$

$A_p = \{ \frac{a}{s}, \; a \in A, \; s \in S \}$

$pA_p = \{ \frac{a}{s}, \; a \in p, \; s \in S \}$.

$\subseteq$: let $x \in qB_p \cap A_p$, so $x = \frac{b}{s} = \frac{a}{t}$, $b \in q$, $a \in A$, $s,t \in S$.

$\exists u \in S$ s.t. $u(bt - as) = 0$ so $A \ni as \cdot u = utb \in q$ so $as \cdot u = utb \in q \cap A$.

$x = \frac{usa}{ust} \in pA_p$.

↳ Recall qB_p is prime id of B_p

qB_p is max of $B_p \Leftrightarrow \underset{pA_p}{\underset{\parallel}{qB_p \cap A_p}}$ is max of A.

which it is, so qB_p is max, $qB_p = q'B_p$.

↳ $q = (qB_p)^c = (q'B_p)^c = q'$. by thm stating $b^{ce} = b$ in localisation.

Proof scheme:

↳ set $p = A \cap q = A \cap q'$ $S = A \setminus p$.

↳ set A_p, $qB_p \subseteq q'B_p$ ideals

↳ show $pA_p = A_p \cap qB_p$

↳ use pA_p max and prev lemma, show qB_p is max

↳ use $(\ )^{ec} = (\ )$ in localisation.

Prop (lying over)

let $A \subset B$ be an integral extension. let p be a prime id of A.

then $\exists\, q$, a prime ideal of B s.t. $q \cap A = p$.

Proof detour

Consider an analogous problem for A_p and B_p. B_p is integral over A_p.

and the ideal pA_p of A_p.

let n be any max id of B_p. Then by thm, $n \cap A_p$ is a max ideal of A_p.

But only max ideal is pA_p. so $n \cap A_p = pA_p$.

commutative diagram

$$\begin{tikzcd} \color{red}{q}\ B \arrow[r, shift left, "e"] & B_p\ \color{red}{n} \arrow[l, shift left, "c"] \\ A \arrow[u, hook] \arrow[r, shift left, "e"] & A_p \arrow[u, hook] \arrow[l, shift left, "c"] \end{tikzcd}$$

$\color{red}{q \cap A}$ $\qquad$ $\color{red}{pA_p}$

as above, n is a max ideal of B_p.

let q be inverse image n of B.

claim that $p = q \cap A$.

WTS: $q \cap A = (pA_p)^c$

① $\boxed{a \in (pA_p)^c} \Leftrightarrow \frac{a}{1} \in pA_p = n \cap A_p \Leftrightarrow \frac{a}{1} \in n$

② $\frac{a}{1} \in n \Leftrightarrow a \in q \Leftrightarrow \boxed{a \in q \cap A}$

so $q \cap A = (pA_p)^c$

using $(\ )^{ec} = (\ )$, so $q \cap A = p$.

Proof scheme:

↳ Consider prob: A_p, B_p and ideal pA_p of A_p. Then let n any max id of B_p. But $n \cap A_p$ is max so $= pA_p$.

↳ consider comm. diagram. get $q = (n)^c$ claim $\overset{(pA_p)^c}{\overset{\shortparallel}{p}} = q \cap A$

↳ two way arrow: $a \in (pA_p)^c \Leftrightarrow \frac{a}{1} \in n$

$a \in q \cap A \Leftrightarrow \frac{a}{1} \in n$

Thm (Going up)

let $A \subset B$ be integral extension of rings.

let $p_1 \subset \cdots \subset p_n$ be prime ideals of A,

let $q_1 \subset \cdots \subset q_m$ be prime ideals of B $m < n$.

and $\forall\ i \le m$, $p_i = q_i \cap A$. Then $\exists$ $q_{m+1} \subset \cdots \subset q_n$ prime ideal of B s.t. $q_m \subset q_{m+1}$ and $q_i \cap A = p_i$ $\forall\ m+1 \le i \le n$

$q_1 \subset \cdots\cdots \subset q_m$ $\color{red}{\subset q_{m+1} \subset \cdots\cdots \subset q_n}$

$p_1 \subset \cdots\cdots \subset p_m \subset p_{m+1} \subset \cdots \subset p_n.$

Pf: suffice to show when $n=2$, $m=1$.

$$\begin{array}{ccc} q_1 & \subset & q_2 \\ \downarrow & & \downarrow \\ p_1 & \subset & p_2 \end{array}$$ want to find such.

Consider the integral extension $A/p_1 \subset B/q_1$

then, p_2/p_1 is a prime ideal of A/p_1.

By lying over, $\exists\ \tilde{q_2}$ ideal of B/q_1 s.t. $\tilde{q_2} \cap A = p_2/p_1$.

let q_2 be the preimage of $\tilde{q_2}$ in B.

so q_2 is a prime ideal of B, $q_1 \subset q_2$, $\tilde{q_2} = q_2/q_1$

and $q_2/q_1 \cap A/p_1 = \tilde{q_2} \cap A/p_1 = p_2/p_1$.

claim $q_2 \cap A = p_2$.

$\supseteq$ if $a \in p_2$, $a + p_1 \in p_2/p_1 \subset q_2/q_1 \Rightarrow a \in q_2 \Rightarrow a \in q_2 \cap A$.

$\subseteq$ let $a \in q_2 \cap A$. $a + q_1 \in q_2/q_1 \cap A/p_1 = p_2/p_1$

so $a = \underset{\in p_2}{a'} + \underset{\in q_1}{b}$ $\quad a - \underset{\in a}{a'} = b \in q_1$ so $b \in A$. so $b \in p_1$

$\Rightarrow a \in p_1 + p_2 \subseteq p_2$.

Proof scheme.

use lying over on

$$\begin{array}{ccc} q_1 & \subset & q_2 \\ \downarrow & & \downarrow \\ p_1 & \subset & p_2 \end{array} \qquad \begin{array}{c} \tilde{q_2} \text{ ideal of } B/q_1 \\ \downarrow \\ p_2/p_1 \text{ ideal of } A/p_1 \end{array}$$ q_2 be preimage of $\tilde{q_2}$

WTS $q_2 \cap A = p_2$ $\left\{ \begin{array}{l} q_2 \text{ is prime} \\ q_2 \cap A = p_2 \\ q_2/q_1 \cap A/p_1 = \tilde{q_2} \cap A/p_1 = p_2/p_1 \end{array} \right.$

Example : not integral extension, going up fails

Consider $\mathbb{Z} \subset \mathbb{Z}[T]$

$\mathbb{Z}[T]$ $(1+2T) \subset \boxed{}$ now if q_2 is ideal of $\mathbb{Z}[T]$, $q_2 \cap \mathbb{Z} = (2)$,

$\mathbb{Z}$ $(0) \subset (2)$ then $2 \in q_2$. $2T \in q_2$. but $(1+2T) - 2T \in q_2$.

q_2 is whole thing have fails.

defn. integral closures

↳ let $A \subset B$ be rings the integral closure of B over A is $\{b \in B \mid b$ integral over $A\}$

↳ let A be an ID. Then int closure of A is its int closure in $\mathrm{Frac}(A)$.

Thm. integral closure is a subring

Proof: $\alpha, \beta \in B$ integral over A. Then $A[\alpha,\beta]$ is finite over A.

also have $(\alpha+\beta)A[\alpha,\beta] \subset A[\alpha,\beta]$

$(\alpha-\beta)A[\alpha,\beta] \subset A[\alpha,\beta]$

$\alpha\beta\, A[\alpha,\beta] \subset A[\alpha,\beta]$

So $A[\alpha,\beta]$ is a module over $A(\alpha+\beta)$, $A(\alpha\beta)$, $A(\alpha-\beta)$. They're faithful

so $\alpha+\beta$, $\alpha\beta$, $\alpha-\beta$ are integral over A.

defn an ID A is integrally closed if $A^{Cl(A)} = A$.

Prop every UFD is integrally closed.

let A be a UFD. let $x \in \mathrm{Frac}(A) \setminus A$.

$x = \frac{a}{b}$. $\exists p$, $p \mid b$, $p \nmid a$. if x is integral over A,

$(a/b)^n + a_{n-1}(a/b)^{n-1} + \cdots + a_1(a/b) + a_0 = 0$

$a^n = -a_{n-1}\cdot a^{n-1} b + \cdots + a_0 b^n$

$\Rightarrow p \mid a^n$, $p \mid a$. $\times$.

Prop let A be an integrally closed ID. let E be a finite extension of $\mathrm{Frac}(A)$. then, $\alpha \in E$ is integral over $A \Leftrightarrow$ its minimal polynomial over $\mathrm{Frac}(A)$ have coefficients in A.

Proof assume $\alpha \in E$ is integral over A. so

$(*)$ $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$ $\quad a_1, \cdots, a_n \in A$.

let $f \in K[T]$ be the min. poly of α over $\mathrm{Frac}(A)$.

L be f's splitting field over K.

$\forall$ root β of f, $\exists$ field iso $\varphi: K[\alpha] \to K[\beta]$ fixing K, $\alpha \mapsto \beta$.

apply φ to $(*)$, we see β is integral over A.

$\Rightarrow$ all roots of f int over A. But coeff of f are

polynomials in its roots. $\Rightarrow$ belong to $A^{Cl(A)}$.
A is int closed, so $f \in A[T]$.

Proof scheme:

- $\hookrightarrow$ If $\alpha \in E$ integral over A, get $(*)\ \alpha^n + \cdots + a_0 = 0$
- $\hookrightarrow$ let f be min poly of α over $\mathrm{Frac}(A)$.
- $\hookrightarrow$ L the splitting field of f over A.
- $\hookrightarrow$ map $\alpha \mapsto \beta$, apply to $(*)$, all roots integral $\Rightarrow$ coefficients int.

(left off page 27)

<u>def element integral over ideal</u>

let $A \subset B$ be rings. let $\mathfrak{a}$ be an ideal of A then an element $b \in B$ is integral $\mathfrak{a}$ if

$$b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = 0 \quad \text{for } a_i \in \mathfrak{a}$$

integral closure of $\mathfrak{a}$ in B is the integral closure of $\mathfrak{a}$ in B.

note: b^m integral over $\mathfrak{a}$ $\rightarrow$ b integral over $\mathfrak{a}$.

<u>Prop equivalent condition for integrality over an ideal</u>

$\longrightarrow$ no proof

$A \subset B$ rings, $\mathfrak{a}$ an ideal of A. Then $b \in B$ is integral over $\mathfrak{a}$ $\Leftrightarrow$ there's an $A[b]$-submodule M of B s.t.

1) M is faithful $A[b]$ module
2) M is finite A-algebra
3) $bM \subset \mathfrak{a}M$.

Prop integral closure of an ideal

let $A \subset B$, $\bar{A}$ be int closure of A in B.

let $\mathfrak{a}$ be ideal of A, then int closure of $\mathfrak{a}$ in B is $\sqrt{\mathfrak{a}\bar{A}}$

Proof $\bar{\mathfrak{a}} = \sqrt{\mathfrak{a}\bar{A}}$

$\subseteq$ let b be integral over $\mathfrak{a}$, then $b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$, $a_i \in \mathfrak{a}$.

$\Rightarrow b \in \bar{A}$ and $b^n = -a_{n-1}b^{n-1} - \cdots - a_1 b - a_0 \in \mathfrak{a}\bar{A} \Rightarrow b \in \sqrt{\mathfrak{a}\bar{A}}$

$\supseteq$ let $b \in \sqrt{\mathfrak{a}\bar{A}}$ then,

$$b^n = \sum_{i=1}^{m} a_i x_i \qquad a_i \in \mathfrak{a}, \quad x_i \in \bar{A}$$

using proposition consider $M = A[x_1, \cdots x_m]$

1) M is a faithful $A[b^n]$ module. $1 \in M$ yes.

2) each x_i finite over A, so M is a finite A-algebra

3) $b^n M \subset \mathfrak{a} M$

b^n integral over $\mathfrak{a} \Rightarrow b$ integral over $\mathfrak{a}$.

Prop. let A be an integrally closed domain E be field, $\text{Frac}\, A \subset E$.
If $x \in E$ is integral over ideal $\mathfrak{a}$ of A, then the min poly of x over $\text{Frac}(A)$ has coefficients in $\sqrt{\mathfrak{a}}$.

lem. A ring, I ideal of A, $S \subset A$ mult set, $I \cap S = \emptyset$.
then there's a max element among ideals of A containing I, disjoint from S. It's also a prime ideal.

proof. ????

Prop prime is contracted of prime iff $p^{ec} = p$

let $\varphi: A \to B$ be ring hom. A prime ideal p of A is the contraction of a prime ideal in B $\Leftrightarrow$ $p^{ec} = p$

$\Rightarrow$ assume $p = q^c$ for q a prime ideal of B. then

$$p^{ec} = q^{cec} = q^c = p$$

$\Leftarrow$ assume $p = p^{ec}$

let $S = A \setminus p$

$\varphi(S)$ is a multiplicative set of B.

↳ $s_1, s_2 \in S$, $\varphi(s_1) \cdot \varphi(s_2) = \varphi(s_1 s_2) \in B$

$\varphi(S)$ disjoint from p^e.

↳ assume not, $\varphi(s_1) \in p^e$. then $s_1 \in p^{ec} = p$ ※.

So $\varphi(S) \cap p^e = \phi$. $\Rightarrow$ $\exists$ max ideal q of B containg p^e disjoint from $\varphi(S)$

q^c is prime ideal of A containg p, disjoint from S.

$\Rightarrow q^c = p \Rightarrow p^{ec} = p$

Thm going down

let $A \subset B$ be integral extension of integral domains, A is integraly closed.

let $p_1 \supset \cdots \supset p_n$ be prime ideals of A and $q_1 \supset \cdots \supset q_m$ be prime ideals of B. $m < n$. and $q_i \cap A = p_i$ for $1 \le i \le m$. Then $\exists$ prime ideals $q_{m+1} \subset \cdots \subset q_n$ s.t. $q_i \cap A = p_i$ and $q_m \supset q_{m+1}$.

$$q_1 \supset q_2 \supset \cdots\cdots \supset q_m \supset q_{m+1} \supset \cdots \supset q_n$$

$$p_1 \supset p_2 \supset \cdots \supset p_m \supset p_{m+1} \supset \cdots \supset p_n$$

Proof: suffice to prove for $n=2, m=1$

have $\begin{cases} q_1 \supset q_2 \\ p_1 \supset p_2 \end{cases}$, $q_1 \cap A = p_1$.

claim: $p_2 = (p_2 B_{q_1}) \cap A$

claim is proven later. Assume claim is true.

$p_2 = (p_2 B_{q_1}) \cap A \Rightarrow p_2 = p_2^{ec}$ so p_2 is a contraction of a prime ideal $\overline{q_2}$ of B_{q_1}. s.t. $\overline{q_2} \cap A = p_2$. ← ????

$$\boxed{\begin{array}{l} B \hookrightarrow B_{q_1} \\ q_2 \leftarrow \overline{q_2} \quad \text{so } q_2 = (\overline{q_2})^c = \overline{q_2} \cap B. \end{array}}$$

Write $q_2 = \overline{q_2} \cap B$ a prime ideal of B.

and $q_2 \cap A = (\overline{q_2} \cap B \cap A) = P_2 \Rightarrow q_2 \cap A = P_2$

By prop " prime ideals of B_{q_1} " $\Leftrightarrow$ "prime ideals of B containing q_1 "

So $q_2 \supset q_1$.

<u>Now, prove the claim.</u>

i.e. $P_2 = (P_2 B q_1) \cap A$.

$P_2 \subset (P_2 B q_1) \cap A$

$\hookrightarrow$ $p \subset p^{ec}$ is true generally

$P_2 \supset (P_2 B q_1) \cap A$

let $a \in P_2 B q_1 \cap A$.

$a \in P_2 B q_1 \Rightarrow a = \frac{y}{s}$ where $y \in P_2 B$ $s \in S = B \setminus q_1$

???

? $\overline{A}$ in B is B, $\Rightarrow$ all elements of $P_2 B$ are integral over P_2.

so $y \in P_2 B$ is integral over P_2.

By prop again, the min poly of y over Frac(A) has cofficients in P_2.

$$y^m + a_{m-1} y^{m-1} + \cdots + a_1 y^1 + a_0 = 0, \qquad a_i \in P_2.$$

$y = as$, $a \in \text{Frac}(A) \Rightarrow$ min poly of s over Frac(A) is

$$(as)^m + a_{m-1}(as)^{m-1} + \cdots + a_1(as) + a_0 = 0$$

dividing by $a^m \Rightarrow$

$$s^m + \frac{a_{m-1}}{a} s^{m-1} + \cdots + \frac{a_1}{a^{m-1}} s + \frac{a_0}{a^m} = 0$$

$s \in B$, so s is integral over A. By prev. prop. all $\frac{a_i}{a^{m-i}} \in A$.

Suppose for ※, that $a \notin P_2$.

Since $\left(\frac{a_i}{a^i}\right) a^i = a_i \in P_2$. have $\frac{a_i}{a^i} \in P_2$.

$\uparrow$ $\notin P_2$

this tells us that $s^m \subset P_2 B$, $P_2 B \subset P_1 B = (q_1 \cap A) B \subset q_1$

so $s^m \in q_1$, So $s \in q_1$ contradiction

Hence $a \in P_2$ so $A \cap P_2 B q_1 \subset P_2$.

Proof scheme (Messy Proof)

1. reduce it to the case $q_1 \supset q_2$, $p_1 \supset p_2$

2. claim $p_2 = (p_2 B_{q_1}) \cap A = p_2^{ec}$

3. If claim is true, get $\overline{q_2}$ of B_{q_1} then q_2 is $(\overline{q_2})^c$

4. show $q_2 \cap A = p_2$ and $q_1 \supset q_2$

5. Show the claim that $p_2 = (p_2 B_{q_1}) \cap A$.

   one direction easy

   other direction: · y integral over p_2

   · y writes as min poly

   · subs $y = as$, $\Rightarrow$ get min poly of s

   · if $a \notin p_2$ get $s \in p_2 B \subset q_1$ ※.

Chapter 9. Dimension theory for f.g. algebras over a field.

def. height and krull dimension

A a ring. height of a prime ideal p of A is maximal d s.t.

$$\underset{\underset{p}{\|}}{p_d} \supsetneq p_{d-1} \supsetneq \cdots \supsetneq p_0$$

(d inclusions and $d+1$ prime ideals are involved in total.)

the krull dimension of ring A is $\sup\{ht(p) \mid p \in \operatorname{Spec}(A)\}$.

$\dim(\{0\}) = -1$

Examples K a field.

1) $\dim K = 0$

2) $\dim K[T_1, \cdots, T_n] = n$

   note $K(T_1, \cdots, T_n) \supsetneq K(T_1, \cdots, T_{n-1}) \supsetneq \cdots \supsetneq K$ implies $\dim \geq n$ But reverse ineq is shown later.

3) an integral domain A is field $\Leftrightarrow$ $\dim(A) = 0$

   $\Rightarrow$ only prime ideal is (0)

   $\Leftarrow$ (0) is the only prime ideal. let $0 \neq x \in A$. x not in max ideal, $\Rightarrow x$ invertible.

4) If A is a PID, either $\dim(A)=0$ or 1.

Show $\dim(A)$ cannot $=2$.

Say $(b) \supsetneq (a) \supsetneq (0)$

then $a \in (b)$ so $a = bx$, $x \in A$. but $b \notin (a)$ a prime so $x \in (a)$ so $x = ay$, $y \in R$

so $a = bay$ $a(1-by)=0$ $by=1$, so b is invertible ※.

Prop. equivalent definitions of transcendental basis:

let $K \subset L$ be fields. A subset A of L is a transcendental basis of L over K

if it satisfies one (hence all of below):

TFAE:

1) A is alg indep over K } Standard
   L is alg over $K(A)$

2) A is alg indep over K } maximally algebraic independent
   $A \cup \{\beta\}$ is not alg indep over K for any $\beta \in L$.

3) L is alg over $K(A)$ } minimally s.t. L is algebraic over $K(A)$.
   but not over $K(A \setminus \{\alpha\})$ for any $\alpha \in A$.

Prop (Properties of transcendental basis)

1) (alg indep set can extend)

let $A \subset L$ be alg indep. Then exists $B \subset L$ s.t. B is tr basis, $A \subset B$.

2) (cardinality) all tr basis of L over K have same cardinality. (denoted $\operatorname{trdeg}_K L$.)

3) (tower law ?)

let $K \subset L \subset E$ let B, C be tr basis of L over K, E over L, respectively,

then $B \cup C$ is a tr. basis of E over K.

$\Rightarrow$ $\operatorname{trdeg}_K E = \operatorname{trdeg}_K L + \operatorname{trdeg}_L E$.

def A an ID, Write $\operatorname{trdeg}_K A = \operatorname{trdeg} \operatorname{Frac}(A)$

Goal If A is a finitely generated K-algebra then $\operatorname{trdeg}_K(A) = \dim(A)$.

def localize at an element

let R be a comm ring and $x \in R$.

then $S_{\{x\}} = \{ x^n (1-rx) \mid n \geq 0,\ r \in R \}$

$S_{\{x\}}$ is a mult set as

$1 \in S_{\{x\}}$ as $x^0 (1-0x) = 1$

mult as $\left(x^{n_1}(1-r_1 x)\right)\left(x^{n_2}(1-r_2 x)\right) = x^{(n_1+n_2)} \left(1-(r_1+r_2-r_1r_2)x\right)$

denote $R_{\{x\}} = S_{\{x\}}^{-1} R$

Prop (about dim of localise at an element)

let R be ring, $n \geq 0$ then

$\dim R \leq n \iff \dim R_{\{x\}} \leq n-1 \quad \forall x \in R.$

Proof 3 observations

1) $m \cap S_{\{x\}} \neq 0 \quad \forall m \in \operatorname{mspec} R,\ x \in R.$

Pf: If $x \in m$, then $x = x^1(1-0\cdot x) \in S_{\{x\}}$ so $x \in m \cap S_{\{x\}}$.

if $x \notin m$ then $x+m$ has an inverse $r+m$ in R/m. So

$1-rx \in S_{\{x\}}$ and $xr+m = (x+m)(r+m) = 1+m \Rightarrow 1-xr \in m.$

2) $p \cap S_{\{x\}} = \emptyset \quad \forall m \supsetneq p$ $m \in \operatorname{mspec} R$, $p \in \operatorname{spec} R$, $x \in m \setminus p$

Pf: for contradiction, say $x^n(1-rx) \in p$, then since $x \notin p$, $1-rx \in p$.

$\Rightarrow$ note $rx \in m$ as $x \in m$ so $1 = 1-rx+rx \in m$ ※.

3) If $x \in R$, we have a bijection

$\{ p \in \operatorname{spec}(R),\ p \cap S_{\{x\}} = \emptyset \} \longleftrightarrow \operatorname{spec} R_{\{x\}}.$

$p \mapsto p^e$

$q^c \leftarrow q$

now prove thm:

$\dim R \leq n \iff \dim R_{\{x\}} \leq n-1 \quad \forall x \in R$

$\Rightarrow$ Suppose $\dim R \leq n$.

Take a chain of distinct prime ideals in $R_{\{x\}}$ of length l.

contract them to R. we get chain of distinct prime ideals of R. each disjant from $R_{\{x\}}$.

By (I), this chain do not include a maximal ideal. So the chain can be extended to a chain of distinct prime ideals of R of length $l+1$. So $l+1 \le \dim R \le n$. so $l \le n-1$ so $\dim R_{\{x\}} \le n-1$.

$\Leftarrow$ assume $\dim R_{\{x\}} \le n-1$, $\forall x \in R$.

If $\dim R \le 0 \Rightarrow \dim R \le n$

So assume $\dim R > 0$. Take a maximal chain of distinct prime ideals of R, of length l. get $m \supsetneq p \supsetneq \cdots$ let $x \in m \setminus p$. Then $p \cap S_{\{x\}} = \emptyset$.

then, removing m, and extend the ps to $R_{\{x\}}$ get a chain of prime ideal of length $l-1$ in $R_{\{x\}}$. So $l-1 \le \dim R_{\{x\}} \le n-1 \Rightarrow l \le n$.

Proof scheme:

WTS $\dim R \le n \Leftrightarrow \dim R_{\{x\}} \le n-1 \quad \forall x \in R$.

lemma: $\hookrightarrow$ $\forall m \in \mathrm{mspec}, x \in R$, $m \cap R_{\{x\}} \neq \emptyset$

$\hookrightarrow$ $\forall m \in \mathrm{mspec}$, $p \in \mathrm{spec}$ $p \subsetneq m$, $x \in m \setminus p$, $p \cap R_{\{x\}} = \emptyset$.

$\Rightarrow$ one side: chain in $R_{\{x\}}$, we can add a max ideal to chain

$\Rightarrow$ chain in R, remove m, extend it, get chain of length $l-1$

<u>Prop. A an ID, K a subfield of A. Then</u>

$$\underline{\mathrm{Trdeg}_K A \geq \dim A.}$$

Proof

Induct on $\mathrm{trdeg}_K A = n$ if $n = \infty$ it's true. So assume n is finite

Induction: If $n=0$ then A is algebraic over K, So A is a field.

$\Rightarrow \dim A = 0$.

let $n \geq 1$. Assume it holds for smaller n.

let $x \in A$. Suffice to show $\dim A_{\{x\}} \le n-1 \Rightarrow \dim A < n = \mathrm{trdeg}_K A$.

note: if $f \in A[T]$ is a polynomial whose lowest coefficient is 1,

i.e. $f = T^l + \sum_{i=l+1}^{m} a_i T^i$ then $f(x) = x^l (1 + x(\text{something})) \in S_{\{x\}}$.

two cases:

1. x is transcendental over k. Then $\operatorname{trdeg}_k k(x) = 1$

so $\underbrace{\operatorname{trdeg}_k k(x)}_{=1} + \operatorname{trdeg}_{k(x)} \operatorname{Frac}(A) = \underbrace{\operatorname{trdeg}_k \operatorname{Frac}(A)}_{=n}$

$\Rightarrow \operatorname{trdeg}_{k(x)} \operatorname{Frac}(A) = n-1$

$\operatorname{Frac}(A) = \operatorname{Frac}(A_{\{x\}})$

so $\operatorname{trdeg}_{k(x)} (\operatorname{Frac} A_{\{x\}}) = n-1$

note $k(x) \subset A_{\{x\}}$ as every element of $k(x)$ can be written as ratio of $k[x]$ with denominator whose lowest nonzero coefficient is 1. So denominator in $S_{\{x\}}$

so $k(x) \in S_{\{x\}}^{-1} A = A_{\{x\}}$.

so $\dim A_{\{x\}} \leq \underbrace{\operatorname{trdeg}_{k(x)} A_{\{x\}}}_{n-1}$ by induction ✓

2. x is algebraic over k.

If x is alg over k, $\exists p \in k[T]$ whose lowest nonzero coefficient is 1, s.t. $p(x) = 0$.

so $0 \in S_{\{x\}}$ so $A_{\{x\}} = 0$ $\Rightarrow$ $\dim A_{\{x\}} = \dim \{0\} = -1 \leq n-1$.

Proof Scheme:

↳ If show $\dim A_{\{x\}} \leq n-1$, we have $\dim A \leq \operatorname{trdeg}_k (A)$.

↳ note: $p(x) \in S_{\{x\}}$ for every p with lowest coeff $=1$

↳ x is tr $/A$

using $\underbrace{\operatorname{trdeg}_k k(x)}_{1} + \underbrace{\operatorname{trdeg}_{k(x)} \operatorname{Frac} A}_{\operatorname{Frac} A_{\{x\}}} = \operatorname{trdeg}_k \operatorname{Frac} A = n$

show that $k(x) \subset A_{\{x\}}$ as denominator in $S_{\{x\}}$.

get that $\dim A_{\{x\}} \leq \operatorname{trdeg}_{k(x)} \operatorname{Frac} A_{\{x\}}$

↳ x is alg over A

$p(x) = 0 \in$ denominator of $A_{\{x\}}$ $\Rightarrow$ $A_{\{x\}} = \{0\}$

Prop Integral extension have the same properties.

let $A \subset B$ be integral extension of rings, then

i) $\dim A = \dim B$

ii) if A, B are ID, k-alg, k field, A a k-subalg of B,

$$\operatorname{trdeg}_k A = \operatorname{trdeg}_k B$$

Proof i) → fill ii later.

to show $\dim A = \dim B$.

↳ show $\dim B \geq \dim A$.

let $n = \dim A$ then get $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$ prime ideals of A. Then by lying over and going up, $\exists\ q_1 \subseteq \cdots \subseteq q_n$ prime ideals of B s.t. $q_i \cap A = p_i$. ($q_i \subsetneq q_{i+1}$, if $q_i = q_{i+1}$, $p_i = q_i \cap A = q_{i+1} \cap A = p_{i+1}$ ※)

so $\dim B \geq n$.

↳ show $\dim B \leq \dim A$.

let $d = \dim B$ and $q_1 \subsetneq q_2 \subsetneq \cdots \subsetneq q_d$ be prime ideal of B.

then $q_1 \cap A \subseteq \cdots \subseteq q_n \cap A$ are prime ideals of A.

If $q_i \cap A = q_{i+1} \cap A$, then by incomparability thm $q_i = q_{i+1}$ ※.

so $q_i \cap A$ are distinct, $\dim(B) \leq \dim(A)$

Proof scheme:

$\dim B \geq \dim A$: going up & lying over

$\dim A \geq \dim B$: incomparability

Prop. $\operatorname{Trdeg} A = \dim_k A$

If A is a f.g. k-alg and an ID, Then

$$\operatorname{Trdeg}_k A = \dim A$$

Proof:

↳ By Noetherian Normalisation thm, $\exists$ B s.t. A is finite over B and that $B = k[x_1, \cdots x_n]$ where $x_1, \cdots x_n$ are alg. indep.

↳ $\dim A = \dim B$ and $\mathrm{trdeg}(A) = \mathrm{trdeg}(B)$

↳ so suffices to show $\dim B = \mathrm{trdeg}(B)$

↳ we know that $\dim B \leq \mathrm{trdeg}_k B$

↳ But then the chain

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots\cdots \subsetneq (x_0, \cdots x_n)$$

shows $\dim B \geq n$ we also know $n = \mathrm{trdeg}_k(B)$.

<u>Example : $A = k[T_1, T_2]$</u> k a field.

let $p \subset A$ be a prime ideal. assume p is not max.

let $f \in p$ and g be irred factor of f. since p prime, $f \in p$.

so $0 \subsetneq (f) \subset p \subsetneq m$ but above thm says $(f) = p$.

so every ideal of A is either (0), or max or generated by an irreducible element. if k alg closed its max ideal is of form $(T_1 - x_1, T_2 - x_2)$. $x_1, x_2 \in k$.

<u>Chapter 10. Nakayama's lemma & applications</u>

<u>thm. Nakayama's lemma</u>

let $\mathfrak{a}$ be an ideal of a ring A s.t. $\mathfrak{a} \subseteq \bigcap_{m \in \mathrm{mspec}(A)} m$.

let M be a f.g. A-module. then

1) $M = \mathfrak{a}M \implies M = 0$

2) if N is an A-submodule of M, s.t. $M = N + \mathfrak{a}M$ then $M = N$.

<u>Proof</u>

1) Suppose towards contradiction $M \neq 0$.

then let M be generated by $e_1, \cdots e_n \in M$ with n minimal. $(n \geq 1)$.

then, $M = \mathfrak{a}M$ implies $e_1 = \sum_{i=1}^n a_i e_i$ $a_i \in \mathfrak{a}$.

so $e_1(\underbrace{1 - a_1}_{\text{unit}}) = \sum_{i=2}^n a_i e_i$

claim $1 - a_1$ does not belong to any max ideal of A. if it did, a_1 also in that max id, then $1 = (1 - a_1) + a_1 \in$ max id ※

So $1-a_1$ is a unit of A. So $e_2, \cdots e_n$ generate M as an A module. (i.e. $e_1(1-a_1)$ can be "produced" by $e_2, \cdots, e_n$) contradicting the minimality.

So $M=0$.

ii) always have $\mathfrak{a}(M/N) = (\mathfrak{a}M+N)/N$

so condition of ii gives $\mathfrak{a}(M/N) = M/N \Rightarrow M/N=0 \Rightarrow M=N$.

Proof scheme:

i) $\mathfrak{a}M=M \Rightarrow M=0$

↳ let $\{e_1, \cdots e_n\}$ be basis

↳ $M=\mathfrak{a}M \Rightarrow e_1 = \sum e_i a_i \Rightarrow (1-a_1)e_1 = \sum_{i=2}^{n} e_i a_i$

↳ $1-a_1$ is a unit $\Rightarrow$ get smaller basis

ii) $\mathfrak{a}(M/N) = (\mathfrak{a}M+N)/N$ But $M=\mathfrak{a}M+N \Rightarrow$ use (i).

Prop. Krull's intersection thm

let A be a Noetherian ring. $\mathfrak{a}$ an ideal of A. $\mathfrak{a}$ is contained in all maximal ideal of A. Then $\bigcap_{n\geq 1} \mathfrak{a}^n = (0)$.

Proof:

claim: $\forall \mathfrak{a} \subseteq A$ ideal, have

$$\underbrace{\bigcap_{n\geq 1} \mathfrak{a}^n}_{M} = \mathfrak{a} \underbrace{\bigcap_{n\geq 1} \mathfrak{a}^n}_{M}$$

M is a f.g. ideal, so Nakayama's lemma implies $\bigcap_{n\geq 1} \mathfrak{a}^n = 0$.

Proof of claim:

$\supseteq$ is clear.

$\subseteq$: A is Noetherian, so let $\mathfrak{a}$ be generated by $\{a_1, \cdots, a_r\} \subseteq A$.

for $n \geq 1$, write H_n = set of homogeneous polynomial of degree n in $A[T_1, \cdots T_n]$.

then, $\mathfrak{a}^n = \{g(a_1, \cdots a_r) \mid g \in H_n\}$ (i.e. degree n. hom poly $=\mathfrak{a}^n$, all extra terms can be in the coefficient).

let

$$S_m = \{f \in H_m \mid f(a_1, \cdots a_n) \in \bigcap_{n\geq 1} \mathfrak{a}^n\}$$

let C be the ideal of $A[T_1, \cdots T_n]$ generated by $\bigcup_{m\geq 1} S_m$

By a corollary to Hilbert's basis theorem, C is generated by a finite subset $\{f_1, \cdots, f_s\}$ of $\bigcup_{m\geq 1} S_m$. let $d_i = \deg f_i$ and $d = \max_i d_i$.

let $b \in \bigcap_{n\geq 1} \mathfrak{a}^n$. so $b \in \mathfrak{a}^{d+1}$. so $b = f(a_1, \cdots a_r)$ for some $f \in H_{d+1}$.

But $f \in S_{d+1} \subset C$ so $(*)$ $f = g_1 f_1 + \cdots + g_s f_s$. for some $g_i \in A[T_1, \cdots, T_r]$.
Since f and each f_i are homogeneous, we replace g_i with its homogeneous part of degree $\deg f - \deg f_i$, and $(*)$ still holds. ??? WHY?

So each g_i is homogeneous of degree $\deg f - \deg f_i = d+1-d_i > 0$.
in particular, constant term of g_i is 0 so $g(a_1, \cdots a_r) \in \mathfrak{a}$.

Finally $b = f(a_1, \cdots a_r) = \sum_{i=1}^{s} \underbrace{g_i(a_1, \cdots a_r)}_{\in \mathfrak{a}} \cdot \underbrace{f_i(a_1, \cdots a_r)}_{\in \bigcap_{i\geq 1} \mathfrak{a}^i} \in \mathfrak{a} \cdot \bigcap_{i\geq 1} \mathfrak{a}^n$.

$a^n = x \cdot f(a_1, \cdots a_r)$ $f \in H_n$

Proof scheme:

- Want to show $\mathfrak{a} \cdot \bigcap_{i\geq 1} \mathfrak{a}^n = \bigcap_{i\geq 1} \mathfrak{a}^n$
- NTS $\bigcap_{i\geq 1} \mathfrak{a}^n \subset \mathfrak{a} \cdot \bigcap_{i\geq 1} \mathfrak{a}^n$.
- $\mathfrak{a}$ is f.g. $\Rightarrow$ $\mathfrak{a} = \text{gen}(\{a_1, \cdots a_r\})$
- $\mathfrak{a}^n = \{g(a_1, \cdots a_r) \mid g \in H_n\}$
- $S_m = \{f \in H_m \mid f(a_1, \cdots a_r) \in \bigcap_{n\geq 1} \mathfrak{a}^n\}$
- C be ideal of $A[T_1, \cdots T_n]$ gen by $\bigcup_{m\geq 1} S_m$
- hilbert basis thm + cor $\Rightarrow$ C generated by $\{f_1, \cdots f_s\}$ of $\bigcup_{m\geq 1} S_m$
- let $b \in \bigcap_{n\geq 1} \mathfrak{a}^n$ $\Rightarrow b \in \mathfrak{a}^{d+1}$ $\Rightarrow$ $b = f(a_1, \cdots a_r)$ for $f \in H_{d+1}$.
- $f \in S_{m+1} \subseteq C$. $f = \sum f_i g_i$ — g_i replaced with hom.
- g_i's constant term is 0. $g_i(a_1, \cdots a_n) \in \mathfrak{a}$ and $f_i(a_1, \cdots a_n) \in \bigcap_{n\geq 1} \mathfrak{a}^n$

Chapter 11. Artinian Rings

def. Artinian Rings

A ring A is artinian if every descending chain of ideals stabilize.

Prop

Artinian $\Leftrightarrow$ every set of ideal has minimal element.

Prop let A be a nonzero artinian ring. Then $\dim(A)=0$.

Proof. note: (0) not always a prime ideal b/c ID is not assumed.

let $p \in \operatorname{Spec} A$. Then $A' = A/p$ is an ID & artinian ring.
then let $0 \neq a \in A'$. Check!

$(a) \supset (a^2) \supset \cdots$ stabilizes at a^n, so $(a^n) = (a^{n+1})$

so $a^n \subset (a^{n+1})$ $a^n = a^{n+1} \cdot b$ for $b \in A'$ since ID, $1 = ab$ so a is a unit. A/p is a field, so p is maximal.

hence $\dim(A) = 0$

Proof idea:

mod prime ideal. then get ID, get every element is a unit.

Examples

1) an ID is art $\Leftrightarrow$ it's a field.
2) every finite ring is Art
3) $k[T]/(T^n)$ is art but $k[T]$ is not
4) $\mathbb{Z}$ is noetherian but not artinian

def. the nil(R)

the nilradical of a ring R is

$nil(R) = \{r \in R \mid r \text{ is nilpotent}\}$

def. nilradical and Jacobson radical.

$$nil(R) = \bigcap_{p \in \operatorname{spec} R} p$$

$$J(R) = \bigcap_{m \in \operatorname{mspec} R} m.$$

Prop. Artinian ring, $J(R) = nil(R)$

Prop. An artinian ring has only finitely many maximal ideals.

Proof.

let Σ be all finite intersection of maximal ideals of R.

let $m_1 \cap \cdots \cap m_n$ be the minimal element.

Claim that $\operatorname{mspec}(R) = \{m_1, \cdots, m_n\}$. Suppose not. Then, $\exists m \in \operatorname{mspec}(A)$ not equal to any m_i. Then, let $a_i \in m_i \setminus m$. Then $a_1 \cdots a_n \in (m_1 \cap \cdots \cap m_n) \setminus m$ as m is prime.

So $m_1 \cap \cdots \cap m_n \cap m \subsetneq m_1 \cap \cdots \cap m_n$ contradicting minimality

Proof idea.

↳ Σ be the set of finite intersection of maximal ideals

↳ Σ has a min element

↳ claim $\operatorname{mspec} A = \{m_1, \cdots m_n\}$.

↳ if not, construct an element in $m_1 \cap \cdots \cap m_n \cap m \subsetneq m_1 \cap \cdots \cap m_n$.

Prop. If A is art, then $(nil(A))^n = 0$ for some $n \geq 1$

proof: consider the chain $(nil(A)) \supset (nil(A))^2 \supset (nil(A))^3 \supset \cdots$

It stabilizes at some n.

We claim $(nil(A))^n = 0$.

Suppose not. let $\Sigma = \{ a \in Id(A) \mid a \cdot (nil(A))^n \neq 0 \}$

then $\Sigma \neq \emptyset$ as $nil(A) \in \Sigma$. So let a be a minimal element of Σ.

So $a\,(nil(A))^n \neq 0$ so let $x \in a$ s.t. $x\, nil(A)^n \neq 0$. then $a = (x)$ by minimality

WTS $x(nil(A)^n) = (x)$.

this shows $(x(nil(A)^n)) = (x)$

↳ $\subseteq$ is clear

↳ $\supseteq$ WTS $nil(A)^n \cdot nil(A)^n \neq 0$ $\boxed{\text{if } nil(A)^n \cdot nil(A)^n \neq 0 \text{ then by minimality it's } x}$

Indeed, $nil(A)^n \cdot nil(A)^n = nil(A)^{2n} \neq 0$.

So $\exists y \in nil(A)^n$ s.t. $xy = x$ for $y \in (nil(A))^n$

$\Rightarrow$ $xy = x$ $\Rightarrow$ $xy \cdot y = xy = x$ $\Rightarrow$ $xy^l = x$ $\forall l \geq 0$, $y \in nil$ so

$x = xy^l = 0$ for large enough l. ※.

Proof scheme:

↳ $nil(A) \supset nil(A)^2 \supset \cdots$ stabilize at n

↳ claim $nil(A)^n = 0$

↳ if not, let $\Sigma =$ set of ideals a, $a\, nil(A)^n \neq 0$ let a be min

↳ let $x \in a$, s.t. $x(nil(x)^n) \neq 0$. then $a = (x)$

↳ show $x \cdot nil(A)^n = (x)$

↳ $xy = x \Rightarrow xy^l = x \Rightarrow x = 0$

def. noetherian modules & artinian modules.

let M be a module over ring A.

↳ M is noe if all chain of submodules $M_1 \subset M_2 \subset \cdots$ stabilizes

↳ Art " " $M_1 \supset M_2 \supset \cdots$ "

Exercise try to prove!!!

1. M is Noe

$\Leftrightarrow$ every A-sub module of M is f.g.

$\Leftrightarrow$ every set $\neq \emptyset$ of submodules has a max element

2. Ring A is Noe/Art $\Leftrightarrow$ it's an Art/Noe as a R-module.

3. let M be a module over A, N an A-submodule.
$\Rightarrow$ M is art/noe
$\Leftrightarrow$ both N and M/N are art/noe.

<u>Prop. let A be a ring s.t. some finite prod of max ideals (not necessary distinct) of A is zero.</u>
then A is Art $\Leftrightarrow$ A is Noe.

Proof: let $m_1, \dots, m_n \in \text{mspec}(A)$ s.t. $m_1 \cdots m_n = 0$.
Consider $A \supset m_1 \supset m_1 m_2 \supset \dots \supset m_1 m_2 \cdots m_n = 0$

where is this used?

let $\begin{cases} M_1 = A/m_1 \\ M_r = m_1 m_2 \cdots m_{r-1} / m_1 \cdots m_{r-1} m_r \quad 2 \le r \le n \end{cases}$

each M_r is an A module & an $\underbrace{A/m_r}_{\text{field}}$ module.

we get bijection:

$\{A/m_r\text{-linear subspaces of } M_r\} \leftrightarrow \{A\text{-submoduls of } A \text{ btwn } m_1 \cdots m_r \text{ and } m_1 \cdots m_{r-1}\}$.

A-submodules of A are just ideals of A.
so if A is artinian (resp. noetherian) then M_r satisfies desc (resp. ascend) on A/m_r-subspaces. so $\dim_{A/m_r} M_r < \infty$. $\Rightarrow$ M_r is both Noe/art $\Rightarrow$ so is R.

not sure entirely.

???

<u>lemma: noetherian ring, every radical ideal is a finite intersection of prime ideals.</u>

<u>example sheet</u>

Thm. a nonzero ring A is Artinian $\Leftrightarrow$ Noetherian with dim 0.

Proof.

$\Rightarrow$ A artinian. dim $A=0$

$\mathrm{Spec}(A) = \mathrm{mspec}(A) = \{m_1, \dots, m_n\}$ $n \geq 1$.

$\mathrm{nil}(A) = \bigcap_{p \in \mathrm{spec}(A)} p$ and $\mathrm{nil}(A)^l = 0$ for some l

so $(m_1 \cap m_2 \cap \cdots \cap m_n)^l = \mathrm{nil}(A)^l = 0$ $l \geq 0$.

So $(m_1 m_2 \cdots m_n)^l = 0$ $\Rightarrow$ noetherian.

$\Leftarrow$??? $\leftarrow$ example sheet!

Chapter 12. Dimension theory for Noetherian rings

def. Exact sequence, short exact sequence.

def. Graded ring

A graded ring is a ring A with a family $(A_n)_{n\geq 0}$ of additive subgroups of A s.t. $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_m A_n \subset A_{m+n}$ $\forall m,n \geq 0$.

Prop A_0 is subring of A and A_m is an A_0-module.

Proof: to show A_0 is subring,

1) multiplicatively closed b/c $A_0 A_0 \subset A_0$

2) $1_A \in A_0$.

Since $A = \bigoplus_{n=0}^{\infty} A_n$,

write $1_A = \sum_{i=0}^{m} y_i$, $y_i \in A_i$

for $z_n \in A_n$

$$\underbrace{z_n}_{\in A_n} = \sum_{i=0}^{m} \underbrace{z_n y_i}_{\in A_{n+i}} \quad \text{so } z_n = y_0 z_n, \text{ so } \forall z,\ y_0 z = z.$$

$$\Rightarrow 1_A = y_0 \in A_0$$

and $A_0 A_m \subset A_m$ $\Rightarrow$ each A_m is an A_0 module.

Example of graded A-module.

$k[T_1, \cdots T_r] = \bigoplus_{n\geq 0}^{\infty} A_n$, each A_n is set of hom polynomial of deg n.

def. Graded A-module

let A be graded ring, $A = \bigoplus_{n\geq 0}^{\infty} A_n$ then a graded A module M is an A-module M, $M = \bigoplus_{n\geq 0}^{\infty} M_n$, M_n an additive subgroup of M, s.t. $A_m M_n \subset M_{m+n}$ $\forall m,n \geq 0$.

since $A_0 M_n \subset M_n$, so M_n is A_0 module.

def. elements in M

an element $x \in M$ is homogenous if $x \in M_n$ for some n.

Any $y \in M$ can be written as $\underbrace{y = \sum_n y_n}_{\text{finite}}$, $y_n \in M_n$, $y_n \neq 0$ are hom. component of y.

def. A homomorphism of graded A-modules

is an A-module homomorphism $f: M = \bigoplus M_n \longrightarrow N \bigoplus N_n$

s.t. $f(M_n) \subset N_n \quad \forall n \geq 0$

def $A_+ = \bigoplus_{i>0} A_i$

Prop. for a ring A, TFAE:

1) A is Noetherian

2) A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof 2) $\to$ 1) finitely gen. alg over noe are noe.

1) $\to$ 2) $A_0 = A/A_+$ is Noetherian. ✓

A_+ is an ideal, generated by the set of all hom elements.

A Noetherian, A_+ is f.g. by $x_1, \cdots x_s$ of hom elements of deg $k_1, \cdots k_s > 0$.

let A' be the A_0-subalg of A generated by $x_1, \cdots x_s$. WTS $A_n \subset A'$ $\forall n \geq 0$.

we argue by induction on n.

↳ $A_0 \subset A'$ ✓

↳ let $y \in A_n$, $n > 0$, $y = \sum_{i=1}^{s} a_i x_i$, $a_i \in A_{n-k_i}$ inductive hypothesis, each

a_i is a polynomial of $x_1, \cdots x_s$ with coeff in A_0. So $y \in A'$.

Proof Scheme:

↳ $A_0 = A/A_+$ is noe

↳ A_+ is an ideal of A generated by hom elements.

↳ A_+ is gen by $x_1, \cdots x_s$, each with degree $k_i > 0$.

↳ $A' = A_0[x_1, \cdots, x_s]$

↳ NTS each $A_n \subset A'$. true by induction & write $y = \sum_i a_i x_i$ $a_i \in A_{\text{smaller}}$.

def let A be a ring. λ an add func on a class $\mathcal{C}$ of A-modules w/ value in $\mathbb{Z}$.

Prop. additive fcn on SES: $0 \to N \to M \to L \to 0$

$M = N \oplus L \Rightarrow \lambda(M) = \lambda(N) + \lambda(L)$.

Prop for the LES of A-modules. $0 \to M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \cdots \to M_n \to 0$

each M_is, ims, kers, in $\mathcal{C}$ then $\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0$

Proof

example sheet

def composition series

a composition series for a module M is chain

$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$ of submodules that cannot be refined.

lemma comp. series have common length & chain can be refined to comp series

If M has a composition series of length n, then n is the length for all composition series. Also, every chain of submodules can be refined to a composition series.

def. $\ell(M)$

Prop M has finite len $\Leftrightarrow$ M is artinian & noetherian.

$\Rightarrow$ clear

$\Leftarrow$ create a composition series.

$M_0 = M$. then repeatedly let M_i be a maximal submodule of M_{i-1}. (exist by Noetherian)

get $M_0 = M \supset M_1 \supset M_2 \supset \cdots$ then terminate b/c artinian

Prop: ℓ is additive ??? prove this!

Hilbert fcns

Setting : $A = \bigoplus_{n \geq 0} A_i$, Noetherian graded ring.

A_0 is noetherian & $A = A_0[x_1, \cdots x_s]$, $x_i \in A_{k_i}$. $k_i > 0$.

let $M = \bigoplus_{n \geq 0} M_i$ f.g. graded A-module. generated by $m_1, \cdots m_t$, $m_i \in M_{r_i}$.

so, each element of M_n (M is an f.g. A module) is of form:

$$\sum_{i=1}^{t} a_i m_i = \sum_{j=1}^{t} f_j(x_1, \cdots, x_s) m_j \quad f \in A_0[T_1, \cdots T_s], \ f_j(x_1, \cdots, x_s) \in A_{n-r_j}$$

M_n is f.g. as A_0 module by all elements of form

$g_j(x_1, \cdots x_n) m_j$, g_j is a monomial in $x_1, \cdots, x_s$ of total degree $n - r_j$

$\Rightarrow M_n$ is f.g. as an A_0 module.

def Poincare series

let λ be a $\mathbb{Z}$-valued additive function on the class of f.g. A_0-modules.

the Poincare series for graded A-module M (w.r.t. λ) is the pow series:

$$P(M,T)=\sum_{n=0}^{\infty}\lambda(M_n)T^n \in \mathbb{Z}[[T]]$$ (pow. serie) where n^{th} coeff is $\lambda(M_n)$.

thm. Hilbert - Serre

$P(M,T)$ is a rational function in T of the form $\dfrac{f(T)}{\prod_{i=1}^{s}(1-T^{k_i})}$ $\quad f(T)\in\mathbb{Z}[T]$

Proof

↳ Recall, A is generated as an A_0-algebra by $x_1,\cdots,x_s$, $x_i\in A_{k_i}$ $k_i>0$

proof is inductio on s.

every element in M is liner comb of A_0 elemets. multiply by A_0 don't change grading. so M is f.g. by $m_1,\cdots m_k$, M have deg bdd by Max (grade (m_i))

↳ $s=0$, $A=A_0$. M is f.g. as an A_0 module. $M_n=0$ for large enough n.

So $P(M,T)$ is a polynomial. ($\lambda(M_n)=0$ for $n>N$, some N).

↳ $s>0$ and thm hold for $s-1$.

Write $M=\bigoplus_{n\in\mathbb{Z}}M_n$, $M_n=0$ $\forall n<0$. let $n\in\mathbb{Z}$.

then the map $\begin{array}{l} M_n \to M_{n+k_s} \\ m \mapsto x_s\cdot m\end{array}$ is a homomorphism of A_0 modules.

get exact sequence of A_0 modules.

$$0\to K_n\to M_n\xrightarrow{\cdot x_s}M_{n+k_s}\to L_{n+k_s}\to 0$$

$K_n=\ker(m\mapsto x_s m)$ $\quad L_{n+k_s}=M_{n+k_s}/\mathrm{im}(m\mapsto x_s m)$ $\quad$ checked exactness at all parts.

let $K=\bigoplus_{n\in\mathbb{Z}}K_n$, $\quad L=\bigoplus_{n\in\mathbb{Z}}L_{n+k_s}$

both of them are A-moduls because $m\in K_n$, $a_i\in A_i$, $i\geq 0$, $a_i m\in M_{n+i}$, as $m\in K_n$, $x_s\cdot a_i\cdot m=a_i x_s m=0$,

so $a_i m\in K_{n+i}$. for L, $L=(\bigoplus_{n\in\mathbb{Z}}M_{n+k_s}/\bigoplus_{n\in\mathbb{Z}}\mathrm{im}(m\mapsto x_s m: M_n\to M_{n+k_s}))$

$\Rightarrow$ K,L are finitely generated A modules (finite b/c K is submodule of M, L is a quotient of M)

↳ both K,L are annihilated by x_s. So both are finitely generated $A_0[x_1,\cdots x_{s-1}]$ modules

why not x_s?

↳ apply λ to the exact sequence,

$$\lambda(K_n)-\lambda(M_n)+\lambda(M_{n+k_s})-\lambda(L_{n+k_s})=0$$

multiply T^{n+k_s} to both sides, get

$$\lambda(M_{n+k_s})\cdot T^{n+k_s}-T^{k_s}\lambda(M_n)\cdot T^n=\lambda(L_{n+k_s})T^{n+k_s}-T^{k_s}\lambda(K_n)T^n$$

Sum over all $n\in\mathbb{Z}$, get

$$(1-T^{k_s})\,P(M,T) = P(L,T) - T^{k_s} P(K,T)$$

apply inductive hypothesis to L, K. ✓ denom is of form $\prod(1-T^{*})$

so numerator is a poly, divide it over to get result.

Proof Scheme:

↳ WTS $P(M,T) = \dfrac{f(T)}{\prod_{j=1}^{s}(1-T^{k_i})}$

↳ let A be generated as an A_0 algebra by $x_0, \cdots, x_s$.

↳ induction. $n=0$, using f.g. module & grading stay same

↳ assume true for $n<s$.

↳ consider map $M_n \to M_{n+k_s}$

↳ get exact sequence of A_0 modules $K = \bigoplus$ kernel, $L = \bigoplus$ quotients

↳ K, L A-moduls, f.g. $A_0[x_0, \cdots x_{s-1}]$???

↳ apply λ to the exact sequence & sum over $\mathbb{Z}$ & apply inductive hypothesis.

Assumptions

λ attain values only in $\mathbb{Z}_{\geq 0}$. $\lambda(N) = 0 \iff N = \{0\}$.

def $d(M)$

Consider $P(M,T) = \dfrac{f(T)}{\prod_{i=1}^{s}(1-T^{k_i})}$, $d(M)$ is the order of pole at $T=1$.

assume $M \neq 0$

Prop. $d(M) \geq 0$ radius of convergence argument.

Pf if $d(M) < 0$ then $\lim_{T \to 1^-} P(M,T) = 0$, $\lambda(M_n) = 0 \;\; \forall n \geq 0 \implies M = 0$

Prop. $d(M/xM) = d(M) - 1$

If $x \in A_k$, $k \geq 0$, is not a zero divisor in M, then $d(M/xM) = d(M) - 1$.

Proof. Consider

$$0 \to K_n \to M_n \xrightarrow{\cdot x_s} M_{n+k_s} \to L_{n+k_s} \to 0$$

replace x_s by $x \in A_k$, $k \geq 0$. Then $K_n = \ker(m \mapsto xm) = 0$

how did this ⊕ work? ???

get $L_n = M_{n+ks} / \mathrm{im}\,(m \mapsto x_s m) = M_{n+k} / x M_n$, So $k=0$, $L = M/xM$

So by $(1-T^{ks})\, P(M,T) = P(L,T) - T^{ks} P(K,T)$

We get $P(M,T) = P(L,T) \,/ (1-T^{ks})$

$= P(M/xM) \,/ (1-T^{ks})$

$d(P(M,T)) = d(\text{RHS}) = d(P(M/xM)) + 1$

So $\underline{d(M) = d(M/xM) + 1}$

Proof Scheme:

↳ Change the LES to $\cdot x$

↳ $K=0$, $L = M/xM$.

↳ substitute summations back.

<u>example that motivates next prop</u>

Consider $k[T_1, \cdots T_n] = \bigoplus_{n \geq 0} A_n$ A_n is additive subgroup consisting of all hom. polynomials of degree n. it's generated as an $A_0 = k$ algebra by $T_1, \cdots T_n \in A_1$.

So $k_1 = \cdots = k_s = 1$ for this choice of generators. (i.e. all T_i in A_1)

<u>Prop. Hilbert polynomial stuff.</u>

Recall k_i is the grading of the i^{th} generator of A as A_0-algebra.

If $k_1 = \cdots = k_s = 1$, then there is a polynomial $HP_M \in \mathbb{Q}[T]$ of degree $d(M) - 1$

s.t. $\lambda(M_n) = HP_M(n)$ for all large enough n.

+ve fcn like ℓ

Proof: Write $d = d(M)$

By the previous prop, $\exists f \in \mathbb{Z}[T]$ s.t. $\lambda(M_n)$ is the coefficient of T^n in $f(T) \cdot \frac{1}{(1-T)^s}$

assume $f(1) \neq 0$, $s = d$ (as $d \geq 0$)

$$\lambda(M_n)\, T^n = \frac{f}{\prod (1-T)^{k_i}}$$

why $(1-T)^s$

why $s = d$

Write $f(T) = \sum_{k=0}^{N} a_k T^k$ $a_k \in \mathbb{Q}$ ($\in \mathbb{Z}$), now

$$(1-T)^{-d} = \sum_{n=0}^{\infty} \binom{d+k-1}{d-1} T^k$$

↳ because $(1-T)^{-1} = \sum_{k=0}^{\infty} T^k$, differentiate both sides $d-1$ times so $\lambda(M_n) = \sum_{k=0}^{N} a_k \underbrace{\binom{d+n-k-1}{d-1}}_{\binom{d+(n-k)-1}{d-1} T^{n-k}} \quad \forall n \geq N$

coefficient $\deg n$ $\qquad a_k T^k$

so $\binom{d+n-k-1}{d-1}$ is a polynomial of n of degree $d-1$.

coeff of $n^{d-1} = \frac{1}{(d-1)!}$ $\qquad$ So $\lambda(M_n)$ is a poly of n in degree $d-1$. coefficient of $n^{d-1} = \sum_{k=0}^{N} a_k/(d-1)! \neq 0$.

Proof Scheme:
- ↳ write $(1-T)^{-d}$
- ↳ write $\lambda(M_n)$ as a sum.
- ↳ $\lambda(M_n)$ as a poly.
- ↳ not sure. what exactly is this polynomial?

???

def Hilbert polynomial

HP_M: sends $\mathbb{Z}$ to $\mathbb{Z}$ but $\in \mathbb{Q}[T]$.

Example. If $A := k[T_1, \dots, T_s]$, then A_n is a k-vector space with basis $\{T_1^{e_1} \cdots T_s^{e_s} \mid \sum e_i = n\}$.

$\dim_k A_n = \binom{s+n-1}{s-1}$ so $P(A,T) = (1-T)^{-s}$ $\quad \lambda(V) = \dim_k V$

unit: Filtrations

def. Filtrations

let M be a module over a ring A.

A filtration $(M_n)_{n=0}^{\infty}$ of M is a sequence of A-modules

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots$$

def $\mathfrak{a}$-filtration, stable $\mathfrak{a}$-filtration

let $\mathfrak{a}$ be an ideal of A. $(M_n)_{n=0}^{\infty}$ is an $\mathfrak{a}$ filtration if

$$\mathfrak{a} M_n \subset M_{n+1} \quad \forall n \geq 0$$

a stable $\mathfrak{a}$-filtration is an $\mathfrak{a}$-filtration s.t.

$$\mathfrak{a} M_n = M_{n+1} \text{ for all large enough } n.$$

Example. $(\mathfrak{a}^n M)_n$ is a stable $\mathfrak{a}$. filtration.

<u>lemma : bounded differance.</u>

let (M_n), (M'_n) be stable $\mathfrak{a}$- filtrations of M. Then for some $n_0 \geq 0$,

we have $\forall n$, $M_{n+n_0} \subset M'_n$, and $M'_{n+n_0} \subset M_n$ $\forall n$.

<u>Proof</u>: statement is transitive can assume $M'_n = \boxed{\mathfrak{a}^n M}$ an stable $\mathfrak{a}$- filtration

Assume that $M'_n = \mathfrak{a}^n M$

show $M'_{n+n_0} \subset M_n$ ✓

note $M'_n = \mathfrak{a}^n M \subset M_n$ since M_n is an $\mathfrak{a}$ filtration $M'_{n+n_0} \subset M'_n \subset M_n$ ✓

show $M_{n+n_0} \subset M'_n$.

Since M_n is stable, $\forall n > n_0$, $\mathfrak{a} M_n = M_{n+1}$

So $M_{n+n_0} = \mathfrak{a}^n M_{n_0} \subset \mathfrak{a}^n M = M'_n$ ✓

<u>Prop. given ideal make graded ring. Given $\mathfrak{a}$.-filtration make graded A^* module.</u>

let A be a ring, $\mathfrak{a}$ an ideal of A, then we can make a graded ring

$$A^* := \bigoplus_{n=0}^{\infty} \mathfrak{a}^n \quad (\mathfrak{a}^0 = A).$$

If M is an A-module, M_n an $\mathfrak{a}$- filtration of M, then

$M^* := \bigoplus_{n=0}^{\infty} M_n$ is a graded A^* module

(since $\mathfrak{a}^m M_n \subset M_{n+m}$, as

if $a \in A^*$, let $a = a_k$ at k^{th} coordinate and 0 elsewhere,

if $m \in M^*$, let $m = m_n$ at n^{th} coordinate and 0 elsewhere

then $am \in \mathfrak{a}^m M_n \subset M_{n+m}$ $= a_k m_n$ at $n+k^{th}$ coord and 0 elsewhere.)

<u>Prop: A noetherian $\Rightarrow$ A^* noetherian</u>

A noetherian $\Rightarrow$ $\mathfrak{a}$ generated by $(x_1, \cdots x_r)$ $\Rightarrow$ A^* generated as an $A[x_1, \cdots x_r]$

algebra $\Rightarrow$ Hilbert's basis thm $\Rightarrow$ A^* Noetherian.

lemma $M^* = $ f.g. A^* module $\Leftrightarrow$ (M_n) stable.

let A be Noetherian. M a finitely generated A-module.

(M_n) an $\mathfrak{a}$-filtration of M. Then TFAE:

1) M^* is a f.g. A^* module.

2) the filtration (M_n) is stable.

Proof: M is a Noetherian A-module as it's a f.g. module over a Noetherian ring A.

$\Rightarrow$ each M_n is f.g.

Let $Q_n := \bigoplus_{r=0}^{n} M_r$ is f.g. A-module. Generally, Q_n is a subgroup of M^*.

Consider the A-submodule M_n^* generated by Q_n. $Q_n \subseteq M^* = \bigoplus M_n$.

then $M_n^* = \underbrace{M_0 \oplus \cdots \oplus M_n \bigoplus_{i=1}^{\infty} \mathfrak{a}^i M_n}_{\text{The } A^*\text{-submodule of } M^* \text{ generated by } Q^n}$ (it being an M^* submodule, it must contain $\mathfrak{a} M_n$, $\mathfrak{a}^n M_n$, etc)

then M_n^* is a f.g. generated module over the Noetherian ring A^* (same finite set that generates Q_n as an A-module) so M_n^* is Noetherian.

then the filtration M_n is stable $\Leftrightarrow$ the ascending chain M_n^* stabilizes.

check!

these two are equal.

$M_n^* = M_0 \oplus \cdots \oplus M_n \oplus \boxed{\mathfrak{a} M_n} \oplus \cdots$

$M_{n+1}^* = M_0 \oplus \cdots \oplus M_n \oplus \boxed{M_{n+1}} \oplus \mathfrak{a} M_{n+1} \oplus \cdots$

now, WTS 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 1).

1) $\Rightarrow$ 2): If M^* is f.g. as an A^* module, then M^* is Noetherian, so (M_n^*) stabilizes so M_n is stable ✓

2) $\Rightarrow$ 1): $M = \bigcup_n M_n^*$ so if (M_n^*) stabilize at some $M_{n_0}^*$, then $M = M_{n_0}^*$. So M^* is f.g. as an A^* module.

Proof scheme:

Prop. (Artin - Rees lemma)

Let $\mathfrak{a}$ be an ideal of a Noetherian ring A.
M a f.g. A-module.
$(M_n)_n$ a stable $\mathfrak{a}$-filtration of M.
If M' is a submodule of M, then $(M' \cap M_n)_n$ is a stable $\mathfrak{a}$-filtration of M'.

Proof

Clearly $(M' \cap M_n) =: K_n$ is a $\mathfrak{a}$-filtration of M'.
then $K = \oplus_n K_n$ is a graded A^* Module and submodule of $M^* = \oplus_n M_n$.
A Noetherian $\Rightarrow$ A^* Noetherian. Note $(M_n)_n$ is a stable $\mathfrak{a}$-filtration,
so by prev lemma, M^* is a f.g. A^*-module. So M^* is Noetherian.
so $\underbrace{K \text{ is finitely generated}}_{\text{as a submodule of } M^*}$ as A^* module. By lemma again, $(K_n)_n$ is stable.

def. The associated graded ring

A a ring, $\mathfrak{a}$ an ideal of A. define:

$$G_{\mathfrak{a}}(A) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1} \qquad (\mathfrak{a}^0 = A)$$

(writing in base p)

this is a graded ring.

Mult is as follows:

If $x \in \mathfrak{a}^n$, $y \in \mathfrak{a}^m$, let $\bar{x} \in \mathfrak{a}^n/\mathfrak{a}^{n+1}$, $\bar{y} \in \mathfrak{a}^m/\mathfrak{a}^{m+1}$ be images,
then $\bar{x} \cdot \bar{y}$ is image of xy in $\mathfrak{a}^{n+m}/\mathfrak{a}^{n+m+1}$.

Check well defined

def. Graded G(A) module.

for an A-module M, an $\mathfrak{a}$-filtration $(M_n)_n$, define

$$G_{\mathfrak{a}}(M) = \bigoplus_{n=0}^{\infty} M_n / M_{n+1} \qquad (\text{since } \mathfrak{a} M_n \subseteq M_{n+1})$$

It's a graded $G(A)$ module. : for $x \in \mathfrak{a}^n$, $m \in M_k$,
let $\bar{x}, \bar{m}$ be image of x, m in $\mathfrak{a}^n/\mathfrak{a}^{n+1}$, M_k/M_{k+1},
then $\bar{x}\bar{m}$ is the image of xm in M_{k+n}/M_{k+n+1}.
Write $G_n(M) = M_n/M_{n+1}$

Prop. Properties about associated A-modules.

let $\mathfrak{a}$ be an ideal of a Noetherian ring A. Then,

1) $G_{\mathfrak{a}}(A)$ is a Noetherian ring.

2) If M is an f.g. A-module and (M_n) a stable $\mathfrak{a}$-filtration of M. then $G(M)$ is a f.g. graded $G_{\mathfrak{a}}(A)$ module.

Proof.

1) A noetherian, $\mathfrak{a}$ is f.g. by $\{x_1, \cdots, x_s\}$. let $\bar{x}_i$ be the image of x_i in $\mathfrak{a}/\mathfrak{a}^2$. Then, $G_{\mathfrak{a}}A = (A/\mathfrak{a}) \oplus \bigoplus_{n=1}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ is generated as an $A/\mathfrak{a}$-algebra by $\bar{x}_1, \cdots \bar{x}_n$. But $A/\mathfrak{a}$ is Noetherian, so by Hilbert basis thm, $G_{\mathfrak{a}}(A)$ is Noetherian.

2) Since M_n stable, take n_0, s.t. $M_{n_0+r} = \mathfrak{a}^r M_{n_0}$ $\forall r \geq 0$.

then $G(M)$ is generated by $\bigoplus_{n \leq n_0} M_n/M_{n+1}$ as a $G_{\mathfrak{a}}(A)$-module.

each M_n/M_{n+1} is a Noetherian A-module. (M is noe $\Rightarrow$ M_n is Noe)

each M_n/M_{n+1} is annihilated by $\mathfrak{a}$. So it's a f.g. $A/\mathfrak{a}$ module. $\Rightarrow$ $G(M)$ is a f.g. $G_{\mathfrak{a}}(A)$-module.

def Primary ideals

an ideal I of a rings A is primary if $I \neq A$ and every zero divisor of I is nilpotent.

Recall & compare the definition of prime, radical, and primary ideals.

Exercise for properties of Primary ideals. 2, 3 are omitted

1) the radical $\sqrt{I}$ of a primary ideal of a ring A is the smallest prime ideal that contain I.
(the map $I \longrightarrow \sqrt{I}$ maps primary ideals to prime ideals, if I is primary, we say I is p primary, $p = \sqrt{I}$)

4) A be a ring

a) $A = \mathbb{Z}$, ideal is prime $\Leftrightarrow$ it's a power of a prime ideal.

b) $m \in \mathrm{mSpec}A$, m^n is m-primary

c) for $p \in \mathrm{Spec}A$, p^n is not necessarily primary, if it is, then it's a p-primary

d) in $K[X, Y]$ not every prime ideal is a power of a prime ideal.

Prop. Three numbers from A (dimension theory for Noetherian local rings)

A a Noe. Local ring with unique max ideal m. for an m-primary ideal q of A, let $\delta(q)$ be the cardinality of the smallest generating set of q. then WTS all below equal.

1) $\dim(A)$ (Krull dim of A)

2) $\delta(A) = \min\{\delta(q) \mid q \text{ is a } m\text{-primary ideal}\}$.

3) $d(G_m A)$ (order of pole at $T=1$ of the rational fcn $P(G_m(A),T) = \sum_{n=0}^{\infty} \ell(m^n/m^{n+1}) \cdot T^n$).

lemma write $\sum_{k=0}^{n-1} p(k)$

let $p \in \mathbb{Q}[T]$, then $\sum_{k=0}^{n-1} p(k) = q(n)$ $\forall n \geq 0$ for some $q \in \mathbb{Q}[T]$ where the leading term of q only depend on the leading term of p.

$\deg q = 1 + \deg p$ ($\deg p = -\infty$ if $p=0$).

Proof since $\sum_{k=0}^{n-1} k^\ell$ is a poly in n of deg $\ell+1$.

$\underbrace{0^\ell + 1^\ell + \cdots + (n-1)^\ell}_{\text{remember: square sum / cubic sum / } 4^{th} \text{ power sum, etc.}}$

Idea: identify two $\mathbb{Q}[T]$

if $f: \mathbb{Z} \to \mathbb{Z}$, is any function if $f(n) = g(n)$ for all large enough, $g \in \mathbb{Q}[T]$, then g is uniquely determined. So we define $\deg f$, leading coeff, leading term of f as g.

Prop. Properties about Noetherian rings

let A be a Noetherian local ring, m its unique maximal ideal, q a m-primary ideal, M a f.g. A-module, (M_n) a q-stable filtration of M then

1) $\ell(M_n/M_{n+1}) < \infty$ $\forall n$

2) for all large enough, n, $\ell(M_n/M_{n+1}) = f(n)$ $\ell(M/M_n) = g(n)$. $f, g \in \mathbb{Q}[T]$

$1 + \deg \ell(M_n/M_{n+1}) = \deg \ell(M/M_n) \leq \delta(q)$

3) the leading terms of $\ell(M_n/M_{n+1})$ and $\ell(M/M_n)$ depend only on A, M, q (not on filtration M_n).

Proof:

i) each M_n/M_{n+1} is a f.g. A/q-module. A/q is artinian b/c it's Noetherian and $\dim A/q = 0$ (no prime ideal between q and m as $\sqrt{q} = m$)

why can't there be an ideal btwn prime & is rad?

so $l(M_n/M_{n+1}) < \infty$ (M_n/M_{n+1} is both noetherian and artinian as a A/q-module)

ii) By prop (Properties of associated A-modules), $G_q(A) = \bigoplus_{n\geq 0} q^n/q^{n+1}$ is a Noetherian graded ring, and $G(M) = \bigoplus_n M_n/M_{n+1}$ is a f.g. graded $G_q(A)$-module.

If $x_1, \dots x_s$ generate q then their image in $\bar{x}_1, \dots, \bar{x}_n$ in q/q^2 generate $G_q(A)$ as an A/q algebra. $\bar{x}_i$ is hom of degree 1.

by prop (if grading of gen = 1, then $\exists$ Hilbert poly), $l(M_n/M_{n+1}) = f(n)$ $\forall n \geq n_0$, $n_0 \geq 0$, some $f \in \mathbb{Q}[T]$. $\deg f \leq s-1$. Then use prev lemma for $f = f$, $g = \sum f$ to get conclusion for ii).

iii) let $(M_n')_n$ be another stable q-filtration of M.

let $g(n) = l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r)$

$g'(n) = l(M'/M_n') = \sum_{r=1}^{n} l(M'_{r-1}/M'_r)$

By ii and the prev lemma, (Σ depend only on ....) for large enough n, $g(n)$, $g(n')$ are poly and depend only on leading term of $l(M_n/M_{n+1})$, $l(M_n'/M_n')$ resp.

(M_n), (M_n') have bounded difference. so $\exists n_0 \geq 0$, $M_{n+n_0} \subset M_n'$, $M_n' \subset M_n$, so $g(n-n_0) \leq g'(n) \leq g(n+n_0)$. Since $g, \tilde{g}$ are poly $\lim \frac{g(n)}{\tilde{g}(n)} = 1$, so they have same leading terms and same is true for $l(M_n/M_{n+1})$, $l(M_n'/M_n')$.

<u>Cor. more cor about deg of M_n/M_{n+1}</u>

A a Noe. loc. ring. m its unique max id. q a m-pri id of A. Then

1) for large enough n, $l(q^n/q^{n+1})$ is a poly of $\deg \leq \delta(q) - 1$.

2) $\deg l(A/q^n) = \deg l(A/m^n)$ and $\deg l(q^n/q^{n+1}) = \deg l(m^n/m^{n+1})$

Pf. 1) follows via $M = A$, $M_n = q^n$

2) $q \subset m$, but also $m^r \subset q$ since in a Noe ring, every ideal contain a power of its radical. Thus, $m^{rn} \subset q^n \subset m^n$ $\forall n \geq 0$.

So $\ell(M/m^n) \le \ell(M/q^n) \le \ell(M/m^{rn})$

so result follows, $n \to \infty$.

Prop for Noe. Loc ring A, $\delta(A) \ge d(G_m(A))$

let q be a m-primary ideal of A generated by $\delta(A)$ elements,

$$\delta(A) = \delta(q) \overset{\text{prev cor}}{\ge} \deg \ell(q^n/q^{n+1}) + 1$$

$$= \deg \ell(\underbrace{m^n/m^{n+1}}_{= M_n}) + 1 = d(G_m(A))$$

polynomial $HP_M \in \mathbb{Q}[T]$, of deg $d(G_m(A)) - 1$ $\ell(M_n) = HP_M(n)$ $\forall n$.

12.24

Prop More about dimension

A Noe, local with max ideal m, $x \in m$ not a zero divisor, then

$$d(G_{m/(x)}(A/(x))) \le d(G_m(A)) - 1$$

Proof:

the map $a \mapsto xa : A \to xA$ is iso of A-modules since x is not a zero divisor.

write $A' = A/(x)$, $m' = m/(x)$. We get a SES of A-modules:

$$0 \longrightarrow xA/(xA \cap m^n) \longrightarrow A/m^n \longrightarrow A'/(m')^n \longrightarrow 0$$

so $\underbrace{\ell(A'/m'^n) = \ell(A/m^n) - \ell(xA/(xA \cap m^n))}_{\text{as polynomials.}}$. The A-modules A and xA are isomorphic. By Artin-Rees $(xA \cap m^n)_n$ is a stable m filtration of xA.

By the prop (with 3 parts. leading part dep on A, m, q, not M_n) The leading part term of $\ell(A/m^n)$ and $\ell(xA/(xA \cap m^n))$ are equal.

So $\underbrace{\deg \ell(A'/m'^n)}_{\deg \ell(m'^n/m'^{n+1})+1} \le \underbrace{\deg \ell(A/m^n) - 1}_{\deg \ell(m^n/m^{n+1})+1}$ same leading coefficient so degree is at least 1 less.

these equalities are 2nd property of the 3 properties.

By prop (HP_M has $\overset{\text{Hilbert poly}}{\deg}$ $d(M)-1$) $d(G_{m'}(A')) \le d(G_m(A)) - 1$

12.25

Prop. more d/dim calculation

for a Noetherian local ring A, whose unique maximal ideal is m, $d(G_m(A)) \ge \dim A$.

Proof:

base

induction on $d(G_m(A))$. If $d(G_m(A)) = 0$, then $\deg \ell(m^n/m^{n+1}) = -1$ (prop 12.11, ℓ has degree $d(\_\_) - 1$). So for large n, $\ell(m^n/m^{n+1}) = 0$ so $m^{n+1} = m^n$.

(m^n is a f.g. A-module, $m \subset J(A) := \bigcap_{n \in m\,\mathrm{spec} A} n$, $m \cdot m^n = 0$) so $m^n = 0$ by Nakayama lemma.

By prop 11.9. (finite prod of max id are 0, Art $\Leftrightarrow$ Noe) so A is Artinian, so $\dim A = 0$.

inductive step

assume that $d(G_m(A)) > 0$. If $\dim(A) = 0$, we done. Assume $\dim A \geq 1$.

take a chain $p_r \supsetneq \cdots \supsetneq p_0$ $r \geq 1$, of prime ids of A.

let $m' = m/p_0$ be the maximal ideal of the Noe. local. I.D. $A' = A/p_0$.

let $x \in p_1 \setminus p_0$. let x' be the image of x in A', then $x' \in m' = m/p_0$ is not a zero divisor

So, by prop 12.24 why is $x \in m$? b/c $x \notin p_0$.

$$d(G_{m'/(x')}(A'/(x'))) \leq d(G_{m'}(A')) - 1 \quad ☆$$

We have a surjective A-module homomorphism $A/m^n \longrightarrow A'/m'^n$ so $l(A'/m'^n) \leq l(A/m^n)$

so $\deg l(A'/m'^n) \leq \deg l(A/m^n)$ why $l \leq l \Rightarrow \deg l \leq \deg l$?

so $\deg l(m'^n/m'^{n+1}) \leq \deg l(m^n/m^{n+1})$ so $d(G_{m'}(A')) \underset{☆☆}{\leq} d(G_m(A))$ $\leftarrow$ l is a polynomial of degree $d-1$ (prop 12.11)

Combing ☆, ☆☆ get

$$d(G_{m'/(x')}(A'/(x'))) \leq d(G_m(A)) - 1$$

By inductive hypothesis,

$\dim A'/(x') \leq d(G_{m'/(x')}(A'/(x'))) \leq d(G_m(A)) - 1$. The images of $\overbrace{p_r, \cdots p_1}^{\text{chain of } r-1}$ in $A'/(x')$

are distinct, so $r-1 \leq d(G_m(A)) - 1$.

$\dim(A)$

Prop. thm relating dim & δ.

for a Noe. local ring (A, m), $\dim(A) \geq \delta(A)$. (i.e. there is an m-primary

ideal q generated by $\dim A$ elements. $\delta(A) = \min(\delta(q) \mid q$ is m primary$)$)

Proof write $d = \dim(A)$. We construct $x_1, \cdots, x_d \in m$ inductively s.t. every prime ideal containing

$(x_1, \cdots, x_i)$ has height $\geq i$. The case $i=0$ is clear.

Assume that $x_1, \cdots, x_{i-1}$ have been constructed. $i-1 < d$. There are only finitely

many prime ideals $p_1, \cdots p_s$, $s \geq 0$, of height $i-1$ containing $(x_1, \cdots, x_{i-1})$

$i-1 < d = \operatorname{ht} m$, so $m \neq p_i$, $\forall i$, so $m \neq \bigcup_i p_i$ by prime avoidance. (ex 2).

let $x_i =$ any $m \setminus \bigcup p_i$

let q be a prime ideal containing $(x_1, \cdots, x_i)$, let p be the minimal among prime ideals containing $(x_1 \cdots x_{i-1})$ and contained in q. If $p \in \{p_1, \cdots p_s\}$ then $q \supsetneq p$ so $ht\, q > ht\, p = i-1$. o.w., $ht\, q \geq ht\, p \geq i$. p not one of P_i so must have height i,

now, if p is a primary ideal containing $I := (x_1, \cdots x_d)$ then $ht\, p \geq d$ so $p = m$

so $\sqrt{I} = m$, so I is m primary.

Thm. 12.28. Dimension thm

For a Noetherian local ring (A, m), $\delta(A) = d(Gm(A)) = \dim A$

Proof

12.23 $\delta(A) \geq d(Gm(A))$

12.25 $d(Gm(A)) \geq \dim A$

12.27 $\dim A \geq \delta(A)$

def. Minimal prime ideal w.r.t. an ideal

I an ideal of A, a min. prime ideal of A is a prime ideal of A, corresponds to a min prime ideal of A/I.

Cor Krull's height thm

let $a = (x_1, \cdots x_r)$ be an ideal of a Noe ring A, then $ht\, p \leq r$ for every minimal prime ideal p of a.

Proof

Consider the localisation map $A \longrightarrow A_p$. Then $\sqrt{a^e}$ is the intersection of prime ideals containing a^e,

??? i.e. $\sqrt{a^e}$ is the unique max ideal p^e of A_p. So a^e is p^e-primary in the local ring (A_p, p^e)

Also, a^e is generated by $\frac{x_1}{1}, \cdots \frac{x_r}{1}$, so $ht\, p = \dim A_p = \delta(A_p) \leq \delta(a^e) \leq r$. ?

$\delta(A_p)$: $\min(\delta(q))$ where q^e is m primary

$\delta(a^e) \leq r$: a^e gen by at most r elements.

Chapter 13. Tensor Products

def. Free A-module over S

A: a ring. S a set.

$A^{\oplus S} = \bigoplus_{s \in S} A \cdot s$ (each element of $A^{\oplus S}$ is a finite sum $\sum_{s \in S'} a_s \cdot s$, $a_s \in A$, $S' \subset S$, $|S'| < \infty$)

def. Tensor Product

M, N are modules over A. Tensor product of M, N is

$M \otimes N = A^{\oplus (M \times N)} / K$ K is submodule of $A^{\oplus (M \times N)}$ generated by union of

1), 2) distributivity in 1^{st}, 2^{nd} coord.

3), 4) scalar mult in 1^{st}, 2^{nd} coord

$1 \cdot (m, n) \in A^{\oplus (M \times N)}$ is denoted $m \otimes n$.

We have a bilinear map: $i_{M \otimes N} : M \times N \to M \otimes N$, $i_{N \otimes N}(m, n) = m \otimes n$.

Prop. Universal property of a tensor product.

M, N are A-modules. Then $(M \otimes N, i_{M \otimes N})$ satisfies the following universal property:

for every A-module L and A-bilinear map $f: M \times N \to L$ there's a unique A-module homomorphism $h: M \otimes N \to L$ s.t. $f = h \circ i_{M \otimes N}$

$$\begin{tikzcd} M \times N \arrow[r, "i_{M \otimes N}"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "h"] \\ & L \end{tikzcd}$$

Proof there's at most one such h as $h(m \otimes n) = f(m, n)$

But such map exists, as $A^{\oplus (M \times N)} \to L$ extending $1 \cdot (m, n) \mapsto f(m, n)$ vanish on all generators of K because of bilinearity. So f factors thru $M \otimes N$, to get $M \otimes N \to L$.

def tensors & pure tensors

tensors are elements of $M \otimes N$

pure tensors are of form $m \otimes n$.

Prop $M \otimes N$, $i_{M\otimes N}$ is the only thing satisfying univ property

for A-modules M, N, if a pair (T, j) T an A-module $j: M\times N \to T$ an A-bilinear map. It satisfies the universal property of prev thm, then there is exactly one A-module isomorphism $\varphi: M\otimes N \to T$ s.t. $\varphi \circ i_{M\otimes N} = j$

(in particular, $M\otimes N \cong T$ as A modules).

Proof: Exercise Try this later.

i.e. if $\exists$ T, j s.t. $\forall f$, $\forall$ bilinear L, f factor thru, then,

$$\begin{tikzcd} M\times N \arrow[r, "j"] \arrow[dr, "f"'] & T \arrow[d] \\ & L \end{tikzcd}$$

j is off by $i_{M\otimes N}$ by an iso and $T \cong M\otimes N$.

Prop sum of tensor $\neq 0$ iff ...

We have $\sum_{i=1}^{l} m_i \otimes n_i \neq 0$ in $M\otimes N$ $\Leftrightarrow$ $\sum_{i=1}^{l} f(m_i, n_i) \neq 0$ for some bilinear map $f: M\times N \to L$ and an A-module L.

Proof

$\Rightarrow$ assume $\sum_{i=1}^{l} m_i \otimes n_i = 0$. let $f: M\times N \to L$ be A-bilinear, L an A-module. Then $f = h \circ i_{M\otimes N}$ for some A-module hom $h: M\otimes N \to L$. So $\sum_{i=1}^{l} f(m_i, n_i) = \sum_{i=1}^{l} h(m_i \otimes n_i) = h\left(\sum_{i=1}^{l} m_i \otimes n_i\right) = 0$

so for all such f, get $\sum f(m_i, n_i) = 0$.

$\Leftarrow$ assume $\sum_{i=1}^{l} m_i \otimes n_i \neq 0$. then $i_{M\otimes N}: M\times N \to M\otimes N$, $\sum_{i=1}^{l} i_{M\otimes N}(m_i, n_i) = \sum_{i=1}^{l} m_i \otimes n_i \neq 0$

Example: embedding $(2\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z})$ in $\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z})$ don't work.

in $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$:

$$2 \otimes (1+2\mathbb{Z}) = 2(1\otimes 1+2\mathbb{Z}) = 1\otimes\left(2(1+2\mathbb{Z})\right) = 1\otimes 2+2\mathbb{Z} = 0$$

in $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$:

$2\otimes(1+2\mathbb{Z})$ doesn't equal 0.

Consider the $\mathbb{Z}$-bilinear map $b: 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$

$$g(2x, y+2\mathbb{Z}) \to xy+2\mathbb{Z}$$

then $g(2, 1+2\mathbb{Z}) \neq 0$ so $2\otimes(1+2\mathbb{Z}) \neq 0$

Prop: if $\sum_{i=1}^{l} m_i \otimes n_i = 0$ in $M \otimes N$, then there are f.g. A-modules $M' \subset M$, $N' \subset N$ s.t. $\sum_{i=1}^{l} m_i \otimes n_i = 0$ in $M' \otimes N'$.

Proof: assume $\sum_{i=1}^{l} m_i \otimes n_i = 0$ in $M \otimes N$, have $\sum_{i=1}^{l} 1 \cdot (m_i, n_i) \in K$. So $\sum_{i=1}^{l} 1 \cdot (m_i, n_i)$ is an A-linear combination of a finite collection of generator of K.

Each generator is a finite A-linear combination of elements of form $(x_j, y_j) \in M \times N$.

let M', N' (resp) be the A-submodules of M, N generated by $\{x_j\}_j$, $\{y_j\}_j$. then $\sum_{i=1}^{l} m_i \otimes n_i = 0$ in $M' \otimes N'$.

Prop. Natural isomorphisms of Tensor products:

1) Commutativity $\quad M \otimes N \to N \otimes M$

$m \otimes n \mapsto n \otimes m$

A-trilinear maps

2) Associativity $\quad (M \otimes N) \otimes P \to M \otimes (N \otimes P) \to M \otimes N \otimes P$

$(m \otimes n) \otimes p \to m \otimes (n \otimes p) \to m \otimes n \otimes p$

3) Distributivity $\quad (\oplus_i M_i) \otimes P \to \oplus_i (M_i \otimes P)$

$(m_i)_i \otimes p \to (m_i \otimes p)_i$

4) Identity element $\quad A \otimes M \mapsto M$

$a \otimes m \mapsto am.$

5) Quotients $\quad M' \subset M$, $N' \subset N$ submodules,

$(M/M') \otimes (N/N') \to (M \otimes N)/L$

$(m + M') \otimes (n + N') \mapsto m \otimes n + L.$

(L is the submodule of $M \otimes N$ generated by $\{m' \otimes n \mid (m', n) \in M' \times N\} \cup \{m \otimes n' \mid (m, n') \in M \times N'\}$)

Example tensor prod of V.S. is a V.S.

V, W are f.d. K vector spaces with bases B, C, resp.

then $V \otimes W$ is a K-vector space with basis $B \otimes C = \{b \otimes c \mid b \in B, c \in C\}$.

Proof exercise.

Def. Extension of Scalars

Restriction of Scalars

$f: A \to B$ ring hom. M is a B-module. Then M is an A-module via $am := f(a)m$ $\forall a \in A$, $m \in M$.

Extension of Scalars

Again $f: A \to B$, also view B as an A-module.

If M is an A-module, $M_B := B \otimes_A M$ (consider both B and M as A-modules).

M_B is a B-module via $b'(b \otimes m) = (b'b) \otimes m$.

Now, verify this using universal property: (Why is this well defined?).

$B \times M \to B \otimes M$

$(b, m) \mapsto b \otimes m$ A-bilinear

By universal property, $h_{b_0}: B \otimes M \to B \otimes M$ is an A-module homomorphism.

$h_{b_0}(b \otimes m) = (b_0 b) \otimes m$

We get $\varphi: B \to \mathrm{End}_{\mathbb{Z}}(B \otimes_A M)$ given by $\varphi(b') = h_{b'}$. φ is a ring hom, since three identities

($h_1 = id$, $h_{b'b''} = h_{b'} h_{b''}$, $h_{b'} + h_{b''} = h_{b'+b''}$) all hold on pure tensors hence hold on $B \otimes_A M$.

Example. (Combining extension of scalars with prop 13.7)

Keep the ring hom $f: A \to B$. Recall A-module generated by set S is $\cong A^{\oplus S}/K$ K an A-submodule of $A^{\oplus S}$. So we study ext of scalars w.r.t. prop 13.7.

(1) $(A^{\oplus S})_B := B \otimes_A A^{\oplus S} \cong (B \otimes_A A)^{\oplus S} \cong (B)^{\oplus S}$

the isomorphism sends $b \otimes v \mapsto b f(v)$, f coordinatewise on $A^{\oplus S}$.

(2) $(A^{\oplus S}/K)_B := B \otimes_A (A^{\oplus S}/K) \cong (B \otimes_A A^{\oplus S})/L \cong B^{\oplus S}/M$.

L is the A-submodules of $B \otimes_A A^{\oplus S}$ generated by $\{b \otimes k \mid (b, k) \in B \times K\}$.

M is the B-submodule of $B^{\oplus S}$ generated by $f(K)$.

the iso sends $b \otimes (v + K) \to b f(v) + M$

(3) If M is generated as an A-module by a subset $S \subseteq A$, then $M_B := B \otimes_A M$ is generated by $\{\varphi(s) \mid s \in S\}$.

Tensor product of algebras

B, C be algebras over a ring A. Consider B, C as A-modules,

We can construct $B \otimes C$. Make it into a ring by $(b \otimes c)(b' \otimes c') = (bb') \otimes (cc')$

and extending linearly.

Fix $(b, c) \in B \times C$, define an A-bilinear map $B \times C \to B \otimes C$ by letting

$(b', c') \mapsto (bb') \otimes (cc')$ (it's bilinear upon checking).

Gives rise to an A-linear map $B \otimes C \to B \otimes C$, $b' \otimes c' \mapsto (bb') \otimes (c'c)$. So mult is well defined.

(Now verify $B \otimes C$ is a ring.)

We can make $B \otimes C$ into a B algebra. via $\begin{array}{l} B \to B \otimes C \\ b \mapsto b \otimes 1 \end{array}$ This gives us two ways

to make $B \otimes C$ into an A-algebra. (The two ways coincide!) (Check!)

Changing the base field of f.g. algebra.

let $K \subset L$ be fields. Consider a f.g. K-algebra $A = K[T_1, \cdots, T_n]/I$. Think of $K[T_1, \cdots T_n]$ as a K vector space.

We get isomorphism of K-vector spaces: $A_L := L \otimes_K A \cong L[T_1, \cdots T_n]/I^e$ where ideal

extension is taken as $K[T_1, \cdots T_n] \to L[T_1, \cdots, T_n]$. The iso sends $x \otimes a \mapsto xa$.

This is a K-vector space isomorphism, but also iso of L-algebras.

Note if $I = (f_1, \cdots f_s)$ in $(K[T_1, \cdots T_n])$ then $I^e = (f_1, \cdots f_s)$ (in $L[T_1, \cdots T_n]$)

So we get an L-algebra isomorphism

$$L \otimes_K (K[T_1, \cdots T_n]/(f_1, \cdots f_s)) \xrightarrow{\sim} L[T_1, \cdots T_n]/(f_1, \cdots f_s)$$

Tensoring homomorphisms

$f: M \to N$, $g: P \to Q$ be A-module homomorphisms.

Then we get A module homomorphism

two steps: 1) well defined via uni prop
2) compositions of homomorphisms also work

$$f \otimes g : M \otimes P \to N \otimes Q$$

$$f \otimes g\,(m \otimes p) \to f(m) \otimes g(p)$$

well defined and extends to a hom on all tensors.

(check!)

Indeed, $(m,n) \mapsto f(m) \otimes g(n)$ is A-bilinear so gives rise to $f\otimes g$.
$M \times M \to P \otimes Q$

Also $(f\otimes g)\circ(h\otimes i) = (f\circ h)\otimes(g\circ i)$ Check by evaluating on pure tensors.

$$(f\otimes g \circ h\otimes i)(a,b) = (f\otimes g)\; h(a)\otimes i(b) = f(h(a))\otimes g(i(b)) = (f\circ h)\otimes(g\circ i)$$

Chapter 14. flat modules.

Prop 14.1. If N is an A-module, then the functor $M \mapsto M\otimes N$ is right exact.

For an exact sequence of A-modules, and an A-module N,

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the following is exact:

$$M'\otimes N \xrightarrow{f\otimes id_N} M\otimes N \xrightarrow{g\otimes id_N} M''\otimes N \longrightarrow 0$$

Proof scheme:
need to show surj @ $M''\otimes N$
exactness @ $M\otimes N$ needs
$M\otimes N/L \longleftrightarrow M''\otimes N$
two sided inverses

Proof:

$g\otimes id_N$ surjective

g is surjective, so $\mathrm{im}(g\otimes id_N)$ contains $\{m''\otimes n \mid (m'',n)\in M''\times N\}$ which is the generating set of $M''\otimes N$ as an A-module. So $g\otimes id_N$ is surjective.

exactness at $M\otimes N$

$g\circ f = 0$, so $(g\otimes id_N)\circ(f\otimes id_N) = 0$ so $\underbrace{\mathrm{Im}(f\otimes id_N)}_{L} \subset \ker(g\otimes id_N)$.

so we get homomorphism

$$(M\otimes N)/L \xrightarrow{\varphi} M''\otimes N \quad \text{satisfying}$$
$$x + L \mapsto (g\otimes id_N)(x).$$

the bilinear map
$$M\times N \to M\otimes N/L$$
$$(m,n) \mapsto m\otimes n + L$$
vanishes on $f(M')\times N$

and gives rise to a bilinear map
$$M''\times N \to M\otimes N/L$$
$$(g(m), n) \mapsto m\otimes n + L.$$

By universal property of $M''\otimes N$ and gives rise to a hom
$$M''\otimes N \xrightarrow{\psi} M\otimes N/L$$
s.t. $$g(m)\otimes n \longmapsto m\otimes n + L$$

now, φ and ψ are two sided inverses $\begin{cases} \varphi\circ\psi : g(m)\otimes n \to m\otimes n + L \to (g\otimes \mathrm{id}_N)(m\otimes n + L) = g(m)\otimes n \\ \psi\circ\varphi : m\otimes n + L \to g(m)\otimes n \to m\otimes n + L \end{cases}$

so if $x \in \ker(g\otimes \mathrm{id}_N)$, $x + L = \psi(\varphi(x+L)) = \psi\big(\underbrace{(g\otimes \mathrm{id}_N)(x)}_{0}\big) = 0 + L$ so $x \in L = \mathrm{im}(f\otimes \mathrm{id}_N)$. ∎

<u>Warning</u> here's an example where $M' \to M \to M''$ is exact but

$$M'\otimes N \to M\otimes N \to M''\otimes N \text{ isn't.}$$

i.e. $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ $f(x) = 2x$, tensor with $N = \mathbb{Z}/2\mathbb{Z}$, get

$$0 \to \mathbb{Z}\otimes(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}\otimes(\mathbb{Z}/2\mathbb{Z}).$$

$$a\otimes(b+2\mathbb{Z}) \mapsto \underbrace{(2a)\otimes(b+2\mathbb{Z})}_{=0}$$

under the isomorphism $\begin{array}{c}\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \\ a\otimes(b+2\mathbb{Z}) \mapsto ab+2\mathbb{Z}\end{array}$ is equiv to $\underbrace{0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}/2\mathbb{Z}}_{\text{not exact!}}$

A very good counterexample.

<u>def. A flat module</u>

a A-module N is flat if $M_1\otimes N \to M_2\otimes N$ is injective whenever $M_1 \to M_2$ is injective.

<u>Example: $f: M_1 \to M_2$ is injective A-module homomorphism.</u>

1) free A-modules are flat: $N_1\otimes A^{\oplus S} \xrightarrow{f\otimes \mathrm{id}} M_2\otimes A^{\oplus S}$ is equivalent to $M_1^{\oplus S} \to M_2^{\oplus S}$ (apply f coord wise, injective)

2) Projective A-modules are flat

Assume that $N_1 \oplus N_2 \cong A^{\oplus S}$.

then, $M_1\otimes(N_1\oplus N_2) \xrightarrow{f\otimes \mathrm{id}_{N_1\oplus N_2}} M_2\otimes(N_1\oplus N_2)$ is injective

this is equivalent to $M_1\otimes N_1 \oplus M_2\otimes N_2 \xrightarrow{f\otimes \mathrm{id}_{N_1}\oplus f\otimes \mathrm{id}_{N_2}} (M_2\otimes N_1)\oplus(M_2\otimes N_2)$

so $f\otimes \mathrm{id}_{N_1}$, $f\otimes \mathrm{id}_{N_2}$ are injective so N_1, N_2 are flat.

?

Subexample: $A = R\times S$, R, S rings, the modules $R\times\{0\}$, $\{0\}\times S$ are flat but not free.

3) Flat modules are torsion free.

If $a \in A$ is not a zero divisor and $0 \neq m \in M$, M a flat A-module

then the map $r \mapsto ar : A \to A$ is injective, and thus so is $m \mapsto am : \underset{M\otimes A}{M} \to \underset{M\otimes A}{M}$.

i.e. $A \xrightarrow{x\mapsto ax} A$ injective

$$\begin{array}{ccc} M\otimes A & \longrightarrow & M\otimes A \\ \wr\wr & & \wr\wr \\ M & \xrightarrow{m\mapsto am} & M \end{array}$$ is injective by flatness.

This is the definition of M being a torsion free A-module. (i.e. M is torsion free if $am=0$ implies either a is a zero divisor in A or $m=0$)

The Tor functor

def. free resolution and the Tor functor

let M and N be A-modules

1) A free resolution of N is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$$

each F_i is a free A-module.

Exercise: every A-module admits a free resolution

$$\cdots \cdots ? \to N \to 0$$

2) $Tor_i^A(M,N)$ is the i^{th} homology group of the chain cx:

$$\to M\otimes F_2 \to M\otimes F_1 \to M\otimes F_0 \to 0$$

↳ delete N from the sequence before tensoring

↳ $F_{-n}=0 \quad \forall\ n\geq 1$

$$Tor_i^A(M,N) = \frac{Ker(M\otimes F_i \to M\otimes F_{i-1})}{Im(M\otimes F_{i+1} \to M\otimes F_i)}$$

Facts: 1) $Tor_i(M,N)$ does not depend on the choice of free resolution.

2) $Tor_i(M,N) \cong Tor_i(N,M)$

3) $Tor_i(M,N)$ can be computed by taking a free resolution of M and tensoring with N.

Example 1 $Tor_0(M,N) = M\otimes N$

$$\to F_1 \to F_0 \to N \to 0$$

$$F_2 \to F_1 \to F_0 \to N \to 0$$

$$M\otimes F_2 \to M\otimes F_1 \to \underbrace{M\otimes F_0 \to M\otimes N}_{\text{surjective}} \to 0$$

this sequence is exact because $M\otimes$ is right exact.

① $$Tor_0(M,N) = \frac{Ker(M\otimes F_0 \to 0)}{Im(M\otimes F_1 \to M\otimes F_0)}$$

$$= \frac{M\otimes F_0}{Ker(M\otimes F_0 \to M\otimes N)} \cong M\otimes N$$

surjective

Example 2 for Tor

Take an A-module N, $x \in A$ not a zero divisor, we have a free resolution:

$$0 \to A \xrightarrow{a \mapsto xa} A \to A/(x) \to 0$$

So $\mathrm{Tor}_i(A/(x), N)$ is the i^{th} homology of

$$0 \to N \xrightarrow{n \mapsto xn} N \to 0$$

thus $\mathrm{Tor}_0(R/(x), N) = N/xN$ $M \otimes N$

$\mathrm{Tor}_1(R/(x), N) = (0 :_N x) := \{n \in N \mid xn = 0\}$. look here

$\mathrm{Tor}_i(R/(x), N) = 0 \quad \forall i \geq 2$.

Prop The snake lemma for Tor

Take a SES,

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r, "f"] & N \arrow[r, "g"] & N'' \arrow[r] & 0 \\
0 \arrow[r] & F_0' \arrow[r] \arrow[u] & F_0 \arrow[r] \arrow[u] & F_0'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_1' \arrow[r] \arrow[u] & F_1 \arrow[r] \arrow[u] & F_1'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_2' \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & F_2'' \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Exercise: Show $\exists \cdots F_2 \to F_1 \to F_0 \to N \to 0$ and maps $F_i' \to F_i \to F_i''$ s.t. the rows are exact & the diagram commutes.

(Hint: use $F_i = F_i' \oplus F_{i}''$)

we tensor with M. (since free modules are flat, so injective, add 0 to left)

$$\begin{tikzcd}
0 \arrow[r] & M \otimes N' \arrow[r] & M \otimes N \arrow[r] & M \otimes N'' \arrow[r] & 0 \\
0 \arrow[r] & M \otimes F_0' \arrow[r] \arrow[u] & M \otimes F_0 \arrow[r] \arrow[u] & M \otimes F_0'' \arrow[r] & 0 \\
0 \arrow[r] & M \otimes F_1' \arrow[r] \arrow[u] & M \otimes F_1 \arrow[r] \arrow[u] & M \otimes F_0'' \arrow[r] \arrow[u] & 0 \\
 & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

we have module homomorphisms (exercise: this is well defined and exact at $\mathrm{Tor}_i(M, N)$).

$$\mathrm{Tor}_i(M, N') \to \mathrm{Tor}_i(M, N) \to \mathrm{Tor}_i(M, N'')$$

$$\| $$

$$\frac{\ker(M \otimes F_i' \to M \otimes F_{i-1}')}{\mathrm{Im}(M \otimes F_{i+1}' \to M \otimes F_i')}$$

we get connecting homomorphisms, $\partial: \mathrm{Tor}_i(M, N'') \longrightarrow \mathrm{Tor}_{i-1}(M, N')$ left, up, left, well defined.

verify yourself?

so we get a LES

$$\underbrace{\longrightarrow \mathrm{Tor}_1(M'', N) \longrightarrow \overbrace{\underbrace{\mathrm{Tor}_0(M', N)}_{M' \otimes N} \longrightarrow \underbrace{\mathrm{Tor}_0(M, N)}_{M \otimes N} \longrightarrow \underbrace{\mathrm{Tor}_0(M'', N)}_{M'' \otimes N} \longrightarrow 0}^{\text{from exactness.}}}_{\text{from LES}}$$

<u>lemma Ideal with Tor</u>

for an ideal I of a ring A,

$$\left(\begin{array}{l} I \otimes M \longrightarrow M \quad \text{is injective} \\ i \otimes m \longmapsto im \end{array} \right) \Leftrightarrow \mathrm{Tor}_1(A/I, M) = 0$$

<u>Proof</u> $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$ has an associated LES

$$\longrightarrow \underset{\overset{\|}{0}}{\mathrm{Tor}_1(A, M)} \longrightarrow \mathrm{Tor}_1(A/I, M) \xrightarrow[\text{injective}]{} \underset{\overset{\|}{I \otimes M}}{\mathrm{Tor}_0(I, M)} \longrightarrow \underset{\overset{\|}{A \otimes M = M}}{\mathrm{Tor}_0(A, M)} \longrightarrow$$

note that $\mathrm{Tor}_1(A, M) = 0$ (as f. res. $\underbrace{0}_{F_1} \to \underbrace{A}_{F_0} \xrightarrow{id} A \to 0$ of A)

thus the map $\mathrm{Tor}_1(A/I, M) \to \mathrm{Tor}_0(I, M)$ is injective. Its image is kernel of $I \otimes M \to M$.

∎

<u>Prop. flatness vs ideals</u>

An A-module M is flat $\Leftrightarrow$ $I \otimes M \to M$ is injective $\forall$ f.g. ideal I of A.

<u>proof</u> take $I \hookrightarrow A$ inclusion of A-modules

$\Rightarrow$ $M \otimes I \hookrightarrow M \otimes A = M$

$\Leftarrow$ Assume that $I \otimes M \to M$ is injective for all f.g. ideal of A.

Claim 1: The natural map $\tau_J : J \otimes_A M \to M$ is injective for any ideal $J \trianglelefteq A$.

let J be an ideal of A. Take $x \in J \otimes M$, $x = \sum_{i=1}^{k} j_i \otimes m_i$ s.t. x maps to 0 in M in $(J \otimes M \to M)$. (i.e. $x \in \ker \tau_J$). so $\sum_i j_i m_i = 0 \in M$. let $I = (j_1, \cdots j_k)$. then $x \in \ker \tau_I$. Since I is f.g., by assumption, $x = 0$, so τ_J is injective.

Now, let $N' \hookrightarrow N$ be an inclusion of A-modules, identifying N' with is image in N, can assume $N' \leq N$. let $\tau : N' \otimes_A M \to N \otimes_A M$ be the natural map.

Claim 2: If N/N' is cyclic (generated by 1 element), then τ is injective

So let $N/N' = xA$, since the map $A \xrightarrow{\cdot x} N/N'$ is surjective, have $N/N' \cong A/J$ for some ideal J of A.

we get SES

$$0 \to N' \to N \to N/N' \to 0$$

we get LES

$$\underset{\substack{\text{IIS} \\ \mathrm{Tor}_1(A/J,\, M) \\ \text{II} \\ 0}}{\mathrm{Tor}_1(N/N', M)} \to \underbrace{\mathrm{Tor}_0(N', M)}_{N' \otimes M} \xrightarrow{\tau} \underbrace{\mathrm{Tor}_0(N, M)}_{N \otimes M}$$

By prev lemma and 1st claim, $J \otimes M \to M$ inj, so this is 0.

therefore, τ is injective by exactness.

Claim 3. If N/N' is finitely generated, then τ is injective.

we have filtration

$$N' = N_0 \leq N_1 \leq \cdots \leq N_m \leq N \quad \text{s.t. } N_i/N_{i-1} \text{ is generated by a single element.}$$

By claim 2, the natural maps $N_i \otimes_A M \to N_{i+1} \otimes_A M$ are all injective. so their composition τ is injective.

now we can prove the result in general. indeed, say

$$x \in \sum_{i=1}^{k} n_i' \otimes m_i \in \ker \tau$$

as in claim 1, restrict τ to the finitely generated module $n_1' A + \cdots + n_k' A$, then claim 3 $\Rightarrow$ $x=0$ so τ is injective. Why restrict to f.g. module of M suffices?

Chapter 13 Discrete Valuation Rings

def. DV

let K be a field, a DV on K is a surjective group homomorphism $v: K^\times \to \mathbb{Z}$ s.t. $v(x+y) \geq \min\{v(x), v(y)\}$ $\forall x,y \in K^\times$, and write $v(0) = \infty$.

def valuation ring of a field.

let v be a discrete valuation on K its discrete valuation ring is $\mathcal{O}_K = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$

note: $\operatorname{Frac} \mathcal{O}_K = K$.

example of a DVR

$p \in \mathbb{Z}$ a prime, $v_p(p^\alpha \frac{m}{n}) = \alpha$, $p \nmid m$, $p \nmid n$.

$A = \mathbb{Z}_{(p)} = \{\frac{m}{n} \mid p \nmid n\}$

Properties of DVR: DVR is a local PID

let A be a DVR of field K w.r.t. valuation v.

units: $x \in A$ is a unit $\Leftrightarrow$ $v(x) = 0$ $\Leftrightarrow$ $v(x^{-1}) = -v(x)$

Associates: for nonzero $x, y \in A$, $v(x) = v(y)$ $\Leftrightarrow$ $v(xy^{-1}) = 0$ $\Leftrightarrow$ xy^{-1} is a unit $\Leftrightarrow$ $(x) = (y)$

since g surjective, $\exists \pi \in A$ s.t. $(\pi) = 1$.

Prop: Any nonzero ideals of A are those generated by π^l, $l > 0$.

Proof: let $\mathfrak{a} \subseteq A$ be a nonzero ideal.

let $l = \min\{v(x) \mid x \in \mathfrak{a}\}$. for find $y \in \mathfrak{a}$ s.t. $v(y) = l$. so $y \sim \pi^l$ so $\pi^l \in \mathfrak{a}$. If $x \in \mathfrak{a}$, $v(x) \geq l$,

so $x \sim \pi^{v(x)} = \pi^{v(x)-l} \pi^l$ so $x \in \pi^l$. Hence $\mathfrak{a} = (\pi^l)$.

so A is a PID with unique max ideal (π) and all ideals are of form (π^l), $l \in \mathbb{Z}_{\geq 0}$.