Part iii Commutative Algebra

Jia Shi

Autumn 2022

1 Week1

2 Week 1 lecture 1.

Definition 2.1:

- k a field
- $A = k[T_1, \ldots, T_n]$ are polynomials in n valuables, called T_i, over k
- take $S \subset A$ $V(S)$ is the vanishing locus of S, which are the points in k^n such that it vanishes in all polynomials in S.
- A set $X \subset K^n$ is an algebraic set if $X = V(S)$ for some $S \subseteq A$.

Remark 1: If I is the ideal of A generated by S, then $V(S) = V(I)$. Fill in why that is later

Remark 2 (Geometric properties vs algebraic properties): So we are trying to construct a dictionary from algebraic properties of ideals with geometric properties of algebraic sets. What geometric properties in particular?

- dimension of the set X
- reducibility of a union $X = X_1 \cup \ldots \cup X_n$ of proper algebraic sets.
- the structure of new point $x \in X$

2.1 Noetherian rings and Hilberts basis theorems

Remark 3 (Motivation for Hilberts basis theorem): Given $A = k[T_1, \ldots, T_n]$ and $S \subseteq A$, possibly infinite. Consider the algebraic set $X = V(S)$. The question is, is there a finite subset $S_0 \subseteq A$ such that $X = V(S_0)$? The answer is yes, as a consequence of the Hilberts basis theorem.

Proposition 2.1 (Three equivalent properties of Noetherian rings):
A ring A is Noetherian if it satisfies one (hence all) of the following properties:

- Every ideal of A is finitely generated
- every ascending chain of ideals of A stabilizes
- Every nonempty set σ of ideals of A has a maximal element.

Proof : fill in later □

Note that every PID is noetherian, but $k[T_1, T_2, \ldots]$ is not Noetherian.

Definition 2.2 ($B-$module M): fill in later

Definition 2.3 (B-algebra A): fill in later

Theorem 2.2 (Hilbert's basis theorem):
Let B be a Noetherian ring. Then every ginitely generated algebra over B is Noethertian.

Proof : Fill in later □

Remark 4 (A is a f.g. B alg): A is a f.g. B algebra $\iff$ A can be written as

$$A = \text{Span}\{a_1^{e_1}, \ldots a_m^{e^m} \mid e_i \geq 0, a_i \in A\}$$

$\iff$ there exists surjective ring homomorphism $B[T_1, \ldots, T_n] \to A$, $T_i \mapsto a_i$.

Proof : fill it in later. The proof should be standard. Fill in this proof has higher priorities □

3 Week 1 lecture 2

Hence as a consequence of Hilbert basis theorem, if k is a field, it is noetherian, so $k[T_1, \ldots, T_n]$ is finitely generated as a K algebra, so every one of its ideal is finitely generated, so we can write $V(S) = V(S_0)$.

Definition 3.1 (finite): Let A be a B algebra. Then we also know that A is a $B-$module. Then A as a B-algebra is finite over B if it is finitely generated as a B-module.
Explicitly, if $S \subseteq A$, then

- S generates A as a $B-$algebra if

$$\text{Span}_B\{s_1^{e_1}, \ldots, s_n^{e^n} \mid e_i \geq 0\} = A$$

- S generates A as a $B-$module if

$$\text{Span}_B\{s_1, \ldots, s_n\} = A$$

Remark 5: It is easier to be generated as an algebra than a module.

Remark 6 (Example of finite generated algebras): Finite diml field extensions are finite generated algebras.
However, $A = K[T, T^{-1}]$ is not finite as a K algebra, also not as a $K[T]$ algebra, but it is finite as a $K[T - T^{-1}]$ algebra, where $A = k[T, T^{-1}] = \text{Span}_{k[T-T^{-1}]}\{1, T\}$.

Definition 3.2 (Integral): Let A be a B algebra. Then $x \in A$ is integral over B is there exists a monic polynomial $p \in B[T]$ such that $p(x) = 0$. A is integral over B is all $x \in A$ is integral over B.
If B is a field, then $x \in A$ is integral over B if and only if x is algebraic over B in the sense of algebraic extensions.

Lemma 3.1 (Ring theory's version of Cramers rule):
Let C be a $n \times n$ matrix over a ring A. Take colum vector $v \in A^n$, such that $Cv = 0$. Then $(\det C)v = 0$.

Proposition 3.2 (Equivalent conditions of being finite):
Let A be a B-algebra. Then TFAE:

- A is a finitely generated integral $B-$algebra.
- A is generated as a B algebra by a finite set of B-integral elements.
- A is finte over B, i.e. A can be finitely generated by some S as a B module.

In general, finite implies finitely generated, but integral does not. For example $\mathbb{Q}$ is integral over $\mathbb{Z}$ but it is not a finitely generated $\mathbb{Z}$ algebra.

Proof : The $1 \to 2, 2 \to 3$ are quite simple. (2 to 3 is like, since αs are algebraic over B, each higher powers of α can be rewritten as lower power.)$3 \to 1$ requires more thoughts, and it requires a lemma.

$\square$

4 Lecture 4

Definition 4.1 (Algebraically independent): Let A be an $k-$algebra where k is a field. Then $x_1, \dots, x_n \in A$ are algebraically independent if the only $p \in k[T_1, \dots, T_n]$ is the zero polynomial. Equivalently, if the $K-$algebra homomorphism $K[T_1, \dots, T_n] \to A$ determined by $T_i \mapsto x_i$ is injective.
In this case, $k[T_1, \dots, T_n]$ is isomorphic to $k[x_1, \dots, x_n]$.

Theorem 4.1 (Noether's normalization theorem):
Let A be a finitely generated algebra over a field K. Then there exists $x_1, \dots, x_n \in A, n \geq 0$, algebraically independent over K, such that A is integral over

$$A' = k[x_1, \dots, x_n]$$

Proof : This proof is a rather lengthy inductive argument.

□

5 Week 2 lecture 1

Theorem 5.1 (Weak Nullstellensatz motivation):
The motivation is that given k a field, then there is a bijection between k^n and $\hom_{k-\text{alg}}(k[T_1, \dots, T_n], k)$. But we can also go from the set $\hom_{k-\text{alg}}(k[T_1, \dots, T_n], k)$ to $\ker f_{\overline{x}} = (T_1 - x_1, \dots, T_n - x_n)$.
However, considering the following:

$$\overline{x} = (x_1, \dots, x_n) \mapsto f_{\overline{x}} \mapsto \ker f_{\overline{x}} = (T_1 - x_1, \dots, T_n - x_n)$$

we must show the equality here

$$\ker f_{\overline{x}} = (T_1 - x_1, \dots, T_n - x_n)$$

Proof : To show the equality, one direction is clear but the other one requires some expanding. □

Remark 7: Why is $[T_1 - x_1, \dots, T_n - x_n]$ a maximal ideal? Because consider the homomorphism

$$f_{\overline{x}} : k[T_1, \dots, T_n] \to k$$

is subjective (check) so

$$k[T_1, \dots, T_n]/\ker(f_{\overline{x}}) \sim k$$

but k is a field so $\ker(f_{\overline{x}}) = (T_1 - x_1, \dots)$ is a max ideal.

So now you get a map

$$k^n \to \text{mspec}(k[T_1, \dots T_n])$$

$$(x_1, \dots, x_m) \mapsto (T_1 - x_1, \dots, T_n - x_n)$$

this is injective because $(x_1, \dots, x_n)$ is the unique point in $V(T_1 - x_1, \dots, T_n - x_n)$. But it is not surjective in general. Later we will learn that it is surjective if K is algebraically closed.

6 Week 2 lecture 2

Recall from last time,
$V : Mspec(L[T_1, \dots, T_n]) \to \{\text{alg subsets of } L^n\}$ is actually injective, where Mspec is the max ideals. Note the image is

- V is injective
- the image of V is the set of all singleton subsets of L^n

Theorem 6.1 (Hilbert's Strong Nullstellensatz):
Let L be an algebraically closed field then $V : Id(L[T_1, \dots, T_n]) \to \{\text{alg subsets of } L^n\}$ is

1. Surjective
2. Not injective anymore

Consider $V((t)) = \{0\}$ and $V((t^2)) = \{0\}$ over $k[T]$.

Theorem 6.2 (Another Strong Nullstellensatz):
Let L be algebraically closed. Then

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

$$V : \{\text{radical ideals of } L(T_1, \dots, T_n)\} \to \{\text{Alg subsets of } L^n\}$$

- This is injective because strong nullstellensatz
- surjective because taking the set $X \subseteq L$, $x = V(\mathfrak{a}) \implies x = V(\sqrt{\mathfrak{a}})$ as radical ideals are more general than ideals

Proposition 6.3 (Zariski's lemma):
Let $K \subseteq L$ be fields, such that L is a finitely generated $k-$algebra. Then $\dim_k L < \infty$.

In other words, if the k algebra $L = k[x_1, \dots, x_n]$ happens to be a field, then $\dim_k L < \infty$.

Remark 8 (Weak Nullstellensatz): Discussion: you can fill in later

7 Week 2 lecture 3

Theorem 7.1 (Weak Nullstellensatz):
for a field K, and a proper ideal $\mathfrak{a}$ of $k[T_1, \dots, T_n]$, then there is a field extension L, $\dim[L : K] < \infty$, $\overline{x} \in L$, such that $f(\overline{x}) = 0, \forall f \in \mathfrak{a}$.

Remark 9: Something about given $p_1, \dots, p_m \in k[T_1, \dots, T_n]$ and existence of $r_1, \dots, r_m \in k[T_1, \dots, T_n]$ such that $\sum r_i p_i = 1$. Then the ps have no common solution in any field extension of k.

Corollary 7.2 (bijection):
If K is alg closed, then the map

$$k^n \to \mathrm{mspec}(k[T_1, \dots, T_n])$$

$$(x_1, \dots, x_n) \mapsto (T_1 - x_1, \dots, T_n - x_n)$$

is a bijection.

Theorem 7.3 (Strong Nullstellensatz):

$$I(V_{K^{cl}}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

Remark 10: In summary, in general, if K is a field, then

$$\{\text{radical ideals of } k[T_1, \dots, T_n]\} \to \{\text{alg subset of } k^n\}$$

we can go from left to right by V and right to left by I.
So $V(I(X)) = X$ so I is injective. And $V(\mathfrak{a}) = V(\sqrt{a})$ means V is surjective.
If K is algebraically closed then $I(V(\mathfrak{a})) = \mathfrak{a}$ for every radical $\mathfrak{a}$. In this case, I, V are bijections.

8 Week 3 lecture 1

Remark 11: Note that V, I are inclusion reversing. i.e.

$$I_1 \subseteq I_2 \implies V(I_1) \supseteq V(I_2)$$

$$X_1 \subseteq X_2 \implies I(X_1) \supseteq I(X_2)$$

If $\mathfrak{c}$ is an ideal of $k[T_1, \dots, T_n]$ then we have bijections between

$$\{\text{radical ideals of } k[T_1, \dots, T_n] \text{contains c}\} = \{\text{radical ideals of } k[T_1, \dots, T_n]/\mathfrak{c}\}$$

$$\longleftrightarrow$$

$$\{\text{alg subsets of } k^n \text{ contained in } \mathfrak{c}\}$$

Definition 8.1 (The zariski topology on k^n): k a field. Then the closed sets are algebraic subsets of k^n. This is a topology.

1. $V((1)) = \varnothing$, $V((0)) = k^n$
2. $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$

3. given $(\mathfrak{a}_i)_{i\in I}$ of $k[T_1, \dots T_k]$, $\bigcap_{i\in I} V(\mathfrak{a}_i) = V(\sum_{i\in I} \mathfrak{a}_i)$

This is called the closed set topology. We write $\mathbb{A}_k^n$ for k^n with the zariski topology.

Definition 8.2 (Open sets in zariski topology): for $f \in k[T_1, \dots, T_n]$

$$D(f) = \{\bar{x} \in \mathbb{A}^n \mid f(\bar{x}) \neq 0\}$$

note that $D(f)$ is a basis for zariski topology.

Remark 12 ($\mathbb{A}_k^n$ is not always hausdorff): it is hausdorff iff k finite.
This also makes $\mathbb{A}_k^n$ irreducible.

We are building more and more in our dictionary relating ideals and algebraic subsets

Example 8.1:

- Every singleton is irred
- a hausdorff space is irred if and only if it is singleton
- $V((pq)) = V(p) \cup V(q)$, p, q irred, and $p \neq cq, \forall c \in k$.

Also note that maximal ideals $\implies$ prime ideals $\implies$ radical ideals.

Lemma 8.2:
if p is a prime ideals of a ring R and $I_1 \cap I_2 \subseteq P$, with I_1, I_2 ideals, then $I_1 \subseteq p$ and $I_2 \subseteq p$.

Proposition 8.3:
K be a field, $X \subseteq \mathbb{A}_k^n$, an algebraic set. Then X is irreducible iff $I(X)$ is prime.
So if $V(\mathfrak{a})$ is irred, then $I(V(\mathfrak{a}))$ is prime. If K is alg closed, then $V(\mathfrak{a})$ irred $\iff \sqrt{\mathfrak{a}}$ is prime.

9 Week3 lecture 2

Remark 13 (Localization): The motivation is that R a ring, $S \subseteq R$ a subset. We want to make a new ring $S^{-1}A$ from A by making elements of S invertible. But we don't want to make the whole ring invertible, we only want to do the minimal necessary work.
Recall this is like in the abelianization of groups, but only abelianizing a small portion.
We can also think of it as fractions where numerators are in R and denominators are in S.

Definition 9.1 (Multiplicative subset):

Definition 9.2 ($S^{-1}A$):

- For $(a_1, s_1), (a_2, s_2) \in A \times S$, we have $(a_1, s_1) \sim (a_2, s_2)$ if $u(a_1 s_2 - a_2 s_1) = 0$ for some $u \in S$.
- $\frac{a}{s}$ is the equivalence class containing the pair (a, s)
- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$, $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 a_2}$
- $S^{-1}A$ is the set of such equivalence classes with $\cdot$ and $+$ as above.

Proposition 9.1:
Addition is well defined.

Now, how do you get from your original A to your localized ring $S^{-1}A$?
Consider the ring homomorphism

$$i_s : A \to S^{-1}A$$

$$a \mapsto \frac{a}{1}$$

Note $\ker i_s = \{a \in A \mid ua = 0, \text{ for some } u \in S\}$.
When is i_s injective, when S has no zero divisors.
If $0 \in S$ then $S^{-1}A = \{0\}$.

Proposition 9.2 (Universal property of $S^{-1}A$):

- $\forall s \in S$, $i_S(s)$ is a unit
- for all rings B, for all ring homomorphism $f : A \to B$ such that $f(s)$ is a unit $\forall s \in S$, there is a unique ring homomorphism satisfying

$$h : S^{-1}A \to B$$

$$h = h \circ i_s$$

  where $f(\frac{a}{s}) = f(a)f^{-1}(s)$.

can add the commutative diagram into this picture.

Proof : The proof just requires to show uniqueness, existence, and well-defined. $\square$

Proposition 9.3:
Let A be a ring, $h \in A$
then

$$A[T]/(1-hT) \xrightarrow{\phi} A_h$$

$$(\sum_{i=1}^{n} a_i T^i) + (1-hT) \mapsto \sum_{i=0}^{n} \frac{a_i}{h_i}$$

is a well defined isomorphism of rings.

Remark 14 (Why localization?): In a ring, ideals are complicated. But after localization, ideals might be easier to study while lots of structures are still preserved about the ring.

Remark 15 (Contraction and extensions of ideals):
`https://crypto.stanford.edu/pbc/notes/commalg/extcon.html`
This gives a very detailed introduction about contraction and extension of ideals.

Definition 9.3 (Contraction and extensions of ideals):
Let $\phi : A \to B$ be a ring homomorphism.
Then

- If $\mathfrak{b}$ is an ideal of B, then $\mathfrak{b}^c = \phi^{-1}(\mathfrak{b})$ is an ideal of A. This is the contraction of $\mathfrak{b}$.
- If $\mathfrak{a}$ is an ideal of A, then $\mathfrak{a}^e$ is the ideals of B generated by $\phi(\mathfrak{a})$.
- if ϕ is surjective then $\phi(\mathfrak{a})$ is an ideal of B.

In general, there is a bijection between

$$\{\text{Contracted ideals of } A\} \longleftrightarrow \{\text{extended ideals of } B\}$$

$$\mathfrak{b}^c \mapsto \mathfrak{b}^{ce}$$

$$\mathfrak{a}^{ec} \leftarrow\!\shortmid \mathfrak{a}^e$$

note that $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ and $b^{ce} \subseteq \mathfrak{b}$.

10 Week 3 Lecture 3

Remark 16 (Extension and contraction w.r.t. localization):
A a ring, $S \subset A$ a multiplicative subset, then let $i_s : A \to S^{-1}A$ where $i_s(a) = \frac{a}{1}$.
Let $\mathfrak{a}$ be an ideal of A and $\mathfrak{b}$ be an ideal of $S^{-1}A$. Then

$$\mathfrak{a}^e = S^{-1}\mathfrak{a} = \{\frac{a}{s} \mid a \in \mathfrak{a}, s \in S\}$$

$$\mathfrak{b}^c = \{a \in A \mid \frac{a}{1} \in \mathfrak{b}\}$$

Note: we know that $1 \in S$ as this is definition of a multiplicative set.

Proposition 10.1:
With A and S as above,

- $\mathfrak{b}^{ce} = \mathfrak{b}$ for every ideal $\mathfrak{b}$ of $S^{-1}A$
- there is a bijection between

$$\{\text{prime ideals of } A \text{ disjoint from } S\} \longleftrightarrow \{\text{prime ideals of } S^{-1}A\}$$

$$\mathfrak{p} \to \mathfrak{p}^e = S^{-1}A$$

$$\mathfrak{q}^e \leftarrow \mathfrak{q}$$

Example 10.2 (Really important example of localization):
Let $\mathfrak{p}$ be a prime ideal of A. Let $S_\mathfrak{p} = A \setminus \mathfrak{p}$ where S is a multiplicative set. Note it is a multiplicative set because its product must not be in $\mathfrak{p}$ by definition. Say $A_\mathfrak{p} = S_\mathfrak{p}^{-1}A$. This gives the above bijection. Then, $S_\mathfrak{p}^{-1}\mathfrak{p}$ contains all prime ideals of $A_\mathfrak{p} = S_\mathfrak{p}^{-1}A$ so it is the unique maximal ideal of $A_\mathfrak{p}$. Hence $A_\mathfrak{p}$ is a local ring: a ring with a unique max ideal. We commonly study a ring A by studying rings of the form $A_\mathfrak{p}$. We can make $\mathbb{Z}_{(p)}$ this way.
Don't quite understand why it contains all prime ideals?

Theorem 10.3 (Going up and going down theorem):
Let $A \subseteq B$ be rings, where $f : A \to B$ inclusion map, then $f^* : \operatorname{spec} A \to \operatorname{spec} B$ by $f^*(\mathfrak{q}) = \mathfrak{q} \cap A$. This theorem deals with finding an ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$ for a given prime idealof A. I.e. extend a prime ideal in A to an ideal of B. It is called $\mathfrak{q}$ lies over $\mathfrak{p}$.

Corollary 10.4:
Let $A \subseteq B$ be rings and be integral extensions, let $\mathfrak{q}$ be a prime ideal of B. Then $\mathfrak{q}$ in B is maximal if and only if $\mathfrak{q} \cap A$ is maximal in$A$.

Proof : Think about this one
Follows from in integral extensions, one is a field iff the other one is. □

Proposition 10.5 (Integral extensions localizaitons are also integral extension):
let $A \subseteq B$ be rings, $A \subseteq S$ a mult set. Then $S^{-1}A \subseteq S^{-1}B$ is also integral extension.

Proposition 10.6 (Lying over):
Let $A \subseteq B$ be an integral extension of rings and let $\mathfrak{p}$ be a prime ideal of A. THen there is a prime ideal q of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.

11 Week 4 lecture 1

Theorem 11.1 (Going up):
$A \subseteq B$ be integral extension. Let $\mathfrak{p}_i, 1 \leq m < n$ be increasing prime ideals of A and $\mathfrak{q}_i$, $1 \leq i \leq m$, be increasing prime ideals of B, with each $\mathfrak{q}$ lie over $\mathfrak{p}$. Then you can extend the $\mathfrak{p}$ ideals, $m+1, \ldots n$ to $\mathfrak{q}$.
This helps us to study the dimension of ideals in the future. The proof idea is to use lying over.

Proposition 11.2 (Incomparability):
Under the context of integral extension and prime ideals. Suppose two ideas $\mathfrak{q}$ in B lie over $\mathfrak{p}$ in A. Then they are the same.

Definition 11.1 (Integrally closed domains): Note that

- $A \subseteq B$ rings, then the integral closure of A in B are elements in B integral over A
- If A is an ID, then the integral closure of A is the integral closure of A in $\text{Frac}(A)$. i.e. elements in $\text{Frac}(A)$ such that it can be written as a root of polynomial in the underlying A.
- An integral domain A is integrally closed if A is the integral closure of A.

Integral closure are interesting and they're completely different from algebraic closure.

Proposition 11.3 (UFD is integrlaly closed.):

11.1 Week 4 lecture 2

Proposition 11.4:
Let A be an integrally closed domain. Let E be finite field extension of $\text{Frac}(A)$.Then $\alpha \in E$ is integral over A if and only if the min poly of α over $\text{Frac}(A) \in A[T]$.

Definition 11.2: $A \subset B$ rings, $\mathfrak{a}$ an ideal of A. Then an element $b \in B$ is integral over $\mathfrak{a}$ if a polynomial in $\mathfrak{a}[x]$ vanish it.

Proposition 11.5:
Let $A \subseteq B$ be rings and let $\mathfrak{a}$ be an ideal of A. Then, $b \in \mathfrak{b}$ is integral over $\mathfrak{a} \iff$ there is an $A[b]$-submodule M of B such that

1. M is a faithful $A[b]$ moduule
2. M is a finite A-algebra
3. $bM \subseteq \mathfrak{a}M$

Proposition 11.6:
Let $A \subseteq B$ be rings. Let $\overline{A}$ be the integral closure of A in B. Let $\mathfrak{a}$ be an ideal of A. Then the integral closure of $\mathfrak{a}$ in B is

$$\sqrt{\mathfrak{a}\overline{A}}$$

Proposition 11.7:
Let A be an integrally closed integral domain. Let E be a field where $\operatorname{Frac}(A) \subseteq E$.
If $x \in E$ is integral over an ideal $\mathfrak{a}$ of A then the coefficients of the minimal poly of x over $\operatorname{Frac}(A)$ are in $\sqrt{\mathfrak{a}}$.

Lemma 11.8:
Let A be a ring. I an ideal of A. $S \subseteq A$ a multiplicative set with $S \cap I = \varnothing$. Then there is a maximal element among the ideals of A containing I and disjoint from S. Any such max element is prime.

Proposition 11.9:
Let $\phi : A \to B$ be a ring hom. A prime ideal P of A is the contraction of a prime ideal of $B \iff p^{ec} = p$.

Theorem 11.10 (Going down):
Let $A \subseteq B$ be integral extensions of IDs, with A integrally closed. Then if you get descending prime ideals of A, $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and partial descending prime ideals of B, $\mathfrak{b}_1, \ldots \mathfrak{b}_m$, $m < n$, such that $\mathfrak{a}_i = \mathfrak{b}_i \cap A$. Then you can extend the $\mathfrak{b}$s down to $\mathfrak{b}_n$.

Proof : long proof ☐

11.2 Week4 lecture 3

12 Dimension theory for finitely generated algebras over a field

12.1 Krull dimension of a ring

Definition 12.1 (Krull dimension): Let A be a ring.

- the height of a prime ideal $\mathfrak{p}$ of A is the maximal length of a chain of prime ideals.

$$\mathfrak{p} = \mathfrak{p}_d \supsetneq \mathfrak{p}_{d-1} \supsetneq \dots p_0$$

  has length d.

- the Krull dimension of a space A is

$$\dim(A) = \sup \{ht(p) \mid p \in \operatorname{spec}(A)\}$$

12.2 Transcendence basis

Definition 12.2 (Transcendence basis): Let $K \subseteq L$ be fields.
A subset A of L is a transcendence basis for L over k if

1. A is algebraically independent over K
2. L is algebraic over $K(A)$

Proposition 12.1 (Three properties of algebraic independence):

1. If $A \subseteq L$ is algebraic independent over K, then $\exists A \subseteq B \subseteq L$ such that B is a transcendental basis for L over K. similar to how you can extend a basis.
2. All transcendental basis for L over K have the same cardinality.
3. For fields, $K \subseteq L \subseteq E$, if b is a tr basis for L/K, and c is a tr basis for E/L, then $B \cup C$ is a tr basis over E/K.

Definition 12.3 (Transcendental degree): The common cardinality of all tr bases for L/K is the transcendental degree of L/K.

12.3 Dimension theory for finitely generated algebras over fields

For an integral domain A which contains field K, $\operatorname{trdeg}_k(A) := \operatorname{trdeg}_k \operatorname{Frac}(A)$.
The goal is to show $\dim(A) = \operatorname{trdeg}_k A$ whenever A is a finitely generated $k-$algebra and an ID.
For a commutative ring R and $x \in R$, define

$$S_{\{x\}} = \{x^n(1 - rx) \mid n \geq 0, r \in R\}$$

this is a multiplicative set. We define $R_{\{x\}} = S_{\{x\}}^{-1} R$.

Proposition 12.2:
Let R be a ring and $n \geq 0$. then

$$\dim R \leq n \iff \dim R_{\{x\}} \leq n-1, \forall x \in R$$

13 Week 5 lecture 1

Recall that our aim is to show $\operatorname{trdeg}_k A = \dim A$ where A is a f.g. $k-$algebra and an ID. Recall we used $S_{\{x\}}$.

Proposition 13.1:
R is a ring, $n \geq 0$, then $\dim R \leq n \iff \dim R_{\{x\}} \leq n-1, \forall x \in R$.

Proposition 13.2:
A an ID, and k- a subfield of A, then

$$\dim A \leq \operatorname{trdeg}_k A$$

Proposition 13.3:
Let $A \subseteq B$ be integral extensions of rings. then

- $\dim A = \dim B$
- if A B are integral domains, as $k-$algebras, where A is a sub-alg of B, then $\operatorname{trdeg}_k A = \operatorname{trdeg}_k B$

13.1 Week 5 Lecture 2

Theorem 13.4:
Let$A$ be a f.g. $k-$algebra where k is a field and an integral domain, then $\dim A = \operatorname{trdeg}_k A$.

Lemma 13.5 (Nakayama's lemma):
Let $\mathfrak{a}$ be an ideal of a ring A such that $\mathfrak{a} \subseteq \bigcap_{\mathfrak{m} \in \text{mspec}} \mathfrak{m}$. Let M be a finite generated A-module. Then

1. if $\mathfrak{a}M = M$ then $M = 0$.
2. if N is a submodule of M such that $M = N + \mathfrak{a}M$ then $M = N$.

Proposition 13.6 (Krull intersection theorem):
Let $\mathfrak{a}$ be an ideal of a noetherian ring A. If

$$\mathfrak{a} \subseteq \bigcap_{\mathfrak{m} \in \operatorname{mspec} A} \mathfrak{m}$$

then

$$\bigcap_{n \geq 1} \mathfrak{a}^n = \{0\}$$

13.2 Week 5 lecture 3

13.3 Artinian rings

Definition 13.1 (Artinian ring): DCC condition

The main goal for this lecture is to show that A is Artinian $\iff$ A is Noetherian and $\dim A = 0$.

Proposition 13.7:
Non-zero Artinian rings have dimension 0, or every prime is maximal.

Proposition 13.8:
For an Art ring A, $|\operatorname{mspec} A| < \infty$.
For an Art ring A, $(nil(A))^n = 0$ for some $n \geq 1$.

Definition 13.2: Let M be a module over a ring A. Then M is said to be Noe/Art if every ascending/descending chain of submodules of M stabilizes.

Proposition 13.9:
Let A be a ring such that some finite product of max ideals of A is 0. Then A is Art $\iff$ A is Noe.

Lemma 13.10:
Let A be a Noe ring. Then every radical ideal of A is a finite intersection of prime ideals. THis is similar to the radical ideals proof.

Theorem 13.11:
Let A be a ring. Then A is artinian $\iff$ A is noetherian and $\dim A = 0$.

13.4 Week 6 Lec 1

13.5 Dimension theory for Noetherian rings

Definition 13.3 (Exact sequence and short exact sequence): We know what they are

Definition 13.4 (Graded rings): A graded ring is $(A, (A_n)_{n=0}^{\infty})$ is a ring A, each $A_n \subseteq A$ are additive subgroups, and A is the direct sum of all the A_ns. We also have $A_n A_m \subseteq A_{m+n}, \forall n, m > 0$. Note A_0 is a subring of A.

Proposition 13.12 (TFAE):

- A is Noetherian
- A_0 is noetherian and A is a f.g. as an A_0-algebra.

Proposition 13.13 (Additive function on classes and A-modules):
Let A be a ring. Let e be a class of A-modules. Let λ be an additive function on e. Then for all short exact sequences $N \to M \to L$, we have $\lambda(M) = \lambda(N) + \lambda(L)$ and $\lambda(M) = \lambda(N) + \lambda(M/N), \forall N \subseteq M$ submodule.

Proposition 13.14:
For an exact sequence, alternating sum of the λ of the modules is 0.

Definition 13.5 (Composition series): Is a descending chain of modules such that each is non-refinable, and the smallest one is 0.

Lemma 13.15 (Composition series's lengths):
If M has a composition series of length n then every composition series of M has length n. This is called the length $l(M)$ for M module.

Proposition 13.16 (Two properties about composition series):

- M has finite length $\iff$ it is artinian and noetherian
- $M \to l(M)$ is an additive function.

Remark 17: Talked about something like, if A is a noetherian graded ring, then if M is a finitely graded A-modules, each component of M is a f.g.A_0-module.

13.6 Week 6 Lec 2

Hilbert functions.

Definition 13.6 (The poincare series): The Poincare series $P(M,T)$ of M (w.r.t. λ) is $P(M,T) = \sum_{n=1}^{\infty} \lambda(M_n) \cdot T^n \in \mathbb{Z}[[T]]$, which is a power series.

Theorem 13.17 (Hilbert-Serre):
$P(M,T)$ is a rational function of the form

$$\frac{f(T)}{\prod_{i=1}^{s}(1-T^{k_i})}$$

We write $d(M)$ for the order of the pole of the rational function $P(M,T)$ at $T=1$. Then $d(M) \geq 0$.

Proposition 13.18:
If $x \in A_k$, $k \geq 0$, is not a zero divisor in M, then $d(M/xM) = d(M) - 1$.

Proposition 13.19:
If $k_1 = \ldots = k_s = 1$, $(A = A_0[x_1, \ldots, x_s], x_i \in A_1)$ then there exists polynomial $HP_m \in \mathbb{Q}[T]$ of degree $d(M)-1$ such that $\lambda(M_n) = HP_m(n)$ for all large enough n.

`https://en.wikipedia.org/wiki/Hilbert_series_and_Hilbert_polynomial`

13.7 Week 6 Lec 3

13.8 Filtrations

Definition 13.7 (Filtrations, $\mathfrak{a}-$filtration, stable $\mathfrak{a}$ filtration): Let M be a module over a ring A. A filtration is a descending sequence $M = M_0 \supseteq M_1 \supseteq \ldots$ of submodules.
If $\mathfrak{a}$ is an idealof A, then $(M_n)_{n=0}^{\infty}$ is an $\mathfrak{a}$ filtration if $\mathfrak{a}M_n \subseteq M_{n+1}, \forall n$. An $\mathfrak{a}-$filtration (M_n) is stable if $\mathfrak{a}M_n = M_{n+1}$ for all large n.

Lemma 13.20 (Bounded difference):
If $(M_n), (M_n)'$ are stable $\mathfrak{a}-$filtrations of M then $\exists n_0 \geq 0$ such that $M_{n+n_0} \subseteq M'_n, M'_{n+n_0} \subseteq M_n, \forall n$.

Proposition 13.21 (Some properties about graded rings):
Let A be a ring, and $\mathfrak{a}$ an ideal such that $A^* = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n$ graded ring and $\mathfrak{a}^0 = A$. If M is an A module and (M_n) a $\mathfrak{a}$-filtration, then $M^* = \bigoplus_n M$ is a graded A^* module. If A is northerian, then $\mathfrak{a}$ is finitely generated. So A^* is finitely generated as an A-algebra. A^* is Noetherian by Hilbert's basis theorem.

Lemma 13.22:
Let A be a noetehrian ring. M a finitely generated A-module. (M_n) an $\mathfrak{a}$-filtration of M. then TFAE:

- M^* is a finitely generated A^* module.
- the filtration (M_n) is stable.

Proposition 13.23 (Artin-Rees lemma):
Let $\mathfrak{a}$ be an ideal of a Noetherian ring A. M a finitely generated $A-$module. $(M_n)_n$ a stable $\mathfrak{a}$-filtration of M. $M' \subseteq M$ an A- submodule. Then $(M_n \cap M')_{n=0}^{\infty}$ is a stable $\mathfrak{a}-$filtration of M'.

13.9 The associated graded ring

Let A be a ring. $\mathfrak{a}$ an ideal of A.

$$G_{\mathfrak{a}}(A) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$$

a graded ring.
If M is an A module, and (M_n) is an $\mathfrak{a}$ filtration, then

$$G(M) = \oplus_n M_n/M_{n+1}$$

is agraded $G_{\mathfrak{a}}(A)$ module.

Proposition 13.24:
Let $\mathfrak{a}$ be an ideal over a Noetherian ring A. then

- $G_{\mathfrak{a}}(A)$ is a noetherian ring
- if M is a finitely generated $A-$module, (M_n) a stable $\mathfrak{a}$-filtration, then $G(M)$ is a finitely generated graded $G_{\mathfrak{a}}(A)$ module.

13.10 Week 7 Lec 1

Definition 13.8 (Primary ideals): An ideal I of R is primary if $I \neq R$ and every zero divisor of R/I is nilpotent.
If I is primary then $\sqrt{I}$ is the small prime ideal that contains I. In particular, $I \to \sqrt{I}$ maps primary ideals to prime ideals. An ideal I is $\mathfrak{p}$ primary is $\mathfrak{p} = \sqrt{I}$.

Consider dimension theory for noetherian local rings. Let $(A, \mathfrak{m})$ be a noetherian local ring. For an $\mathfrak{m}$-primary ideal q, $\delta(q)$ is the cardinality of the minimal generating set of q.

Proposition 13.25:
There are three numbers form A.

- $\dim A$
- $\delta(A) = \min\{\delta(q) \mid q \text{ is a } \mathfrak{m}\text{primary ideal of } A\}$
- $d(G_{\mathfrak{m}}(A))$ is the order of the pole at $T = 1$ fo the rational function $\sum_{n=1}^{\infty} l(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \cdot T^n$.

These three numbers are all equal

Proposition 13.26:
Let $(A, \mathfrak{m})$ be a noetherian local ring. $\mathfrak{q}$ an $\mathfrak{m}-$primary ideal of A. M a f.g. $A-$module. (M_n) a $\mathfrak{q}-$stable filtration of M.then

- $l(M_n/M_{n+1}) < \infty$
- $l(M_n/M_{n+1}) = f(n)$, $l(M/M_n) = g(n)$, $f, g \in \mathbb{Q}[t]$ for large enough n. $1 + \deg l(M_n/M_{n+1}) = \deg l(M/M_n) \leq \delta q$.
- the leading terms of $l(M_n/M_{n+1})$ and $l(M/M_n)$ depend only on $A, \mathfrak{m}, q$ but not on (M_n).

Corollary 13.27:
If $(A, \mathfrak{m})$ is a noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$ primary ideal, then

- for large enough n $l(\mathfrak{q}^n/\mathfrak{q}^{n+1})$ is a polynomial of degree $\leq \delta(\mathfrak{q}) - 1$.
- $\deg l(\mathfrak{q}/\mathfrak{q}^n) = \deg(A/\mathfrak{m}^n)$ and $\deg l(\mathfrak{q}^n/\mathfrak{q}^{n+1}) = \deg l(\mathfrak{m}^n/\mathfrak{m}^{n+1})$.

Proposition 13.28:
For a noetherain local ring, $\delta(A) \geq d(G_{\mathfrak{m}}(A))$

Proposition 13.29:
For $(A, \mathfrak{m})$ noetherian local, if $x \in \mathfrak{m}$, not a 0 divisor, then

$$d(G_{\mathfrak{m}/(x)}(A/(x))) \leq d(G_{\mathfrak{m}}(A)) - 1.$$

13.11 Week 7 Lec 2

Proposition 13.30:
Let $(A, \mathfrak{m})$ be a Noetherian local ring, $x \in \mathfrak{m}$, not a zero divisor. Then

$$d(G_{\mathfrak{m}/(x)}(A/(x))) \leq d(G_{\mathfrak{m}}(A)) - 1$$

Proposition 13.31 ((Heart of the whole theorem)):
For a Noetherian local ring $(A, \mathfrak{m})$, $d(G_{\mathfrak{m}}(A)) \geq \dim(A)$.

Corollary 13.32:
For every Noetherian local ring A, we have $\dim A \geq \delta(A)$.

Theorem 13.33 (Big theorem):
$(A, \mathfrak{m})$ is Noetherian local ring. then

$$\dim(A) = \delta(A) = d(G_{\mathfrak{m}}(A))$$

Corollary 13.34 (Krull's height theorem):
A is a noetherian ring. $x_1, \dots, x_r \in A$, then every minimal prime ideal for $(x_1, \dots, x_r)$ has height $\leq r$.

13.12 Week 7 Lecture 3

14 Tensor products and flatness

Let M, N be $A-$modules. Then we define $M \otimes N$ to be the finite sums of elements of the form $m_i \otimes n_i$.
Also recall the definition of bilinear maps and $A^{\oplus S}$ where A is a ring and S is an arbitrary set.
The formal definition of tensor product is as follows:

Definition 14.1: Let M, N be A-modules. Then tensor product $M \otimes N$ is $A^{\oplus M \times N}/K$ where K is the A-submodule of $A^{\oplus M \times N}$ generated by

- $\{(m, n_1) + (m, n_2) - (m, n_1 + n_2) \mid m \in M, n_1, n_2 \in N\}$
- $\{(m_1, n) + (m_2, n) - (m_1 + m_2, n) \mid m_1, m_2 \in M, n \in N\}$
- $\{a(m, n) - (am, n) \mid a \in A, m \in M, n \in N\}$
- $\{a(m, n) - (m, an) \mid a \in A, m \in M, n \in N\}$

The image of $1 \cdot (m, n) \in A^{\oplus (M \times N)}/K$ is denoted $m \otimes n$.
We get a bilinear map

$$i_{M \otimes N} : M \times N \to M \otimes N$$
$$(m, n) \mapsto m \otimes n$$

Proposition 14.1 (The universal property of tensor product):
For $A-$modules M, N, the pair $(M \otimes N), i_{M \otimes N} : M \times N \to M \otimes N$ has the following property: for every A-module L,, and a bilinear map $f : M \otimes N \to L$, there exists a unique A-module homomorphism $h : M \otimes N \to L$ such that $f = h \circ i_{M \otimes N}$.

$$\begin{tikzcd} M \times N \arrow[r, "i_{M\otimes N}"] \arrow[dr, "f\text{:bilinear}"'] & M \otimes N \arrow[d, dotted, "\exists! h, A-\text{linear}"] \\ & L \end{tikzcd}$$

Proposition 14.2:
The pair $(M \otimes N, i_{M\otimes N})$ is uniquely determined by the universal property.

Proposition 14.3:
We have $\sum_{i=1}^{l} m_i \otimes n_i \neq 0 \iff \sum_{i=1}^{l} f(m_i, n_i) \neq 0$ for some A-bilinear map $f : M \times N \to L, L$ an A-module.

Proposition 14.4:
If $\sum m_i \otimes n_i = 0(*)$ in $M \otimes N$ then there are f.g. $A-$submodules, $M' \subseteq M, N' \subseteq N$, such that $(*)$ holds in $M' \otimes N'$.

Proposition 14.5:
$\otimes$ is commutative, associative, distributive, and have an identity. (In terms of the tensor spaces created by this operation is isomorphic.)

Definition 14.2 (Quotients): for submodules $M' \subseteq M, N' \subseteq N$, $M/M' \otimes N/N' = M \otimes N/L$. Then L is generated by $, m' \otimes n, m \otimes n'$.

14.1 Week 8 Lec 1

Remark 18 (Restriction of scalars): If $f : A \to B$ a ring hom, M- a B-module is an A module via $a \cdot m = f(a)m$.

Remark 19 (Extension of scalars): If N is an A module then $N_B := B \otimes N$ is a B module, where B as an A module. then $b_0(b \otimes n) = (b_0 b) \otimes n$. Get a map $B \times N \to B \otimes N$, where A bilinear. Using universal property, we get $h_{b_0} : B \otimes N \to B \otimes N$, $h_{b_0}(b \otimes n) = (b_0 b) \otimes n$. The map $B \to \mathrm{End}_{\mathbb{Z}}(B \otimes M)$, $b_0 \mapsto h_{b_0}$ is a ring homomorphism. Get $\mathbb{C} \otimes (\mathbb{R}^n) \simeq \mathbb{C}^n$.

Remark 20 (Tensor product of algebras): If B, C are algebras over A, then $B \otimes_A C$ becomes a ring. Operations are well-defined.

Remark 21 (Tensoring homomorphisms): Let $f : M \to N, g : P \to Q$ be $A-$module homomorphisms. Then we can define $f \otimes g : M \otimes P \to N \otimes Q$.

14.2 Flat modules

Proposition 14.6 (Tensoring an exact sequence):
For an exact sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

of A modules, for an A-module N, the sequence

$$M' \otimes N \xrightarrow{f \otimes id_N} M \otimes N \xrightarrow{g \otimes id_N} M'' \otimes N \to 0$$

is exact

But there's an warning: if the following is exact,

$$M' \to M \to M''$$

yet

$$M' \otimes N \to M \otimes N \to M'' \otimes N$$

might not be.

Definition 14.3 (Flat module): An A-module N is flat if $M_1 \otimes N \xrightarrow{f \otimes id_N} M_2 \otimes N$ is injective whenever $M_1 \xrightarrow{f} M_2$ is.

14.3 Week 8 Lecture 2

Note that $N_1 \oplus N_2$, A-modules, is free, then both N_1, and N_2 are flat.

Proposition 14.7 (Flat modules are torsion free):
Consider $M \otimes A$, if M is flat then $\forall a \in A, m \in M$, if $am = 0$ then either a is a zero divisor or $m = 0$.

Definition 14.4 (The Tor functor): Let M, N be A modules. Then

- A free resolution of N is an exact sequence

  $$\ldots \to F_2 \to F_1 \to F_0 \to N \to 0$$

  where F_i is a free A-module. Note it always exists.

- $\mathrm{Tor}_i^A(M, N)$ is the ith homology of chain complex. or $f_i \circ f_{i+1} = 0$.

  $$\ldots \to M \otimes F_2 \xrightarrow{f_2} M \otimes F_1 \xrightarrow{f_1} M \otimes F_0 \to 0$$

  Note that you no longer get N after tensoring.

Here is a list of facts:

- $\text{Tor}_i(M, N)$ doesn't depend on the choice of free resolution
- $\text{Tor}_i(M, N) \simeq \text{Tor}_i(N, M)$
- $\text{Tor}_i(M, N)$ can also be computed by taking a free resolution of M and tensoring with N.

We also have a few commutative diagram propositions similar to the snake lemma and there's one where you can fill out groups. I.e. we use snake lemma to get some LES between $\text{Tor}_i(M, N)$, $\text{Tor}_i(M, N')$,$\text{Tor}_i(M, N'')$.

Lemma 14.8:
For an ideal I of a ring A,
$$I \otimes M \to M$$
$$i \otimes m \to im$$
is injective, $\iff$ $\text{Tor}_1(A/I, M) = 0$.

Proposition 14.9:
An A-modules M is flat $\iff$ $I \otimes M \to M$ is injective for every f.g. ideal I of A.

Proposition 14.10:
An A-module is flat $\iff$ $I \otimes M \to M$ is injective forall f.g. ideal I of A.

Proof : There are three cases. The prof himself got confused in case 3. You should look at it after. □

Definition 14.5 (DVRs): Discrete Valuation Rings

Remark 22 (A few remarks about discrete valuation rings):
For nonzero items $x, y \in A$, $v(x) = v(y) \iff xy^{-1}$ is a unit.
The only nonzero ideal of A are generated by $(\pi^l), l \geq 0$.
The ideals are
$$(0) \subseteq (\pi^0) \subseteq (\pi^1) \subseteq (\pi^2) \subseteq \dots$$