1 Homotopy stuff

Lecture 1

- Def: two maps $f_0, f_1 : X \to Y$ are homotopic
- Lem: homotopic maps is an $\sim$ and if $f_0 \sim f_1, g_0 \sim g_1$, then $f_0 \circ g_0 \sim f_1 \circ g_1$
- Def: $[X, Y] = \text{maps}(X, Y)/\sim$
- Prop: $[X, \mathbb{R}^n]$ has one element.
- Def: a space X is contractible
- Prop: a space Y is contractible $\iff$ $[X, Y]$ has 1 element for all spaces X.
- Def: two spaces, X, Y are homotopic equivalent
- Def: pair of spaces, map of pairs, two map of pairs are homotopic
- Def: homotopy groups
- Rmk: properties of homotopy groups
- Prop: $\pi_n(X, p)$ is a group. (Addition, identity map, abelian for $n > 1$, and inverses)
- Def: map of pairs induces maps between homotopy: $f : (X, p) \to (Y, q)$ induces $f_* : \pi(X, p) \to \pi(Y, q), [\phi] \mapsto [f \circ \phi]$
- **Prop: functionality of the above map f_***
- **Prop: homotopy invariance: homotopic maps induces the same maps on homotopy groups.**

Lecture 2

- Def: the $n-$simplex Δ^n
- Def: Faces of the $n-$simplex, f_I where $I \subset \{0, \ldots, n\}$
- Def: Face maps: $F_I : \Delta^{|I|-1} \to f_I \subset \Delta^n$
- Def: Chain complex, (C, d)
- **Def: the ith homology group**
- Def: $x \in \ker$ then x is closed or a cycle. $x \in \text{im}$, then x is exact/boundary. if $dx = 0$, write $[x]$ be its image in $H_*(C)$.
- Def: the chain complex of the $n-$simplex, $(S_*(\Delta^n), d)$
- **Prop: for the chain complex above,** $d^2 = 0$
- Def: the reduced chain complex of Δ^n. $(\tilde{S}_*(\Delta^n), d)$
- Rmk: the idea behind reduced chain complex
- Def: the singular chain complex of X. Denoted as $(C_*(X), d)$. Elements of $C_k(X)$. The differential d.
- Def: $\phi_\sigma : S_*(\Delta^k) \to C_*(X)$

- Prop: $d \circ \phi_\sigma = \phi_\sigma \circ d$
- Prop: $d^2 = 0$ in $C_k(X)$
- Def: singular homology on $X.H_i(C_*(X))$
- Prop: computing $H_*(\{\bullet\})$.

Lecture 3

- Def: reduced singular chain complex
- Prop: If X is path connected, then $H_0(X) = \mathbb{Z}$
- Def: a subcomplex (A, d) of a chain complex (X, d)
- Prop: Two properties of a subcomplex: If (A, d) is a subcomplex of (C, d) then (A, d) is a chain complex and $(C/A, d)$ is a chain complex.
- Def: the quotient complex
- Prop: if $A \subset X$, then $C_*(A)$ is a subcomplex of $C_*(X)$.
- Def: singular chain complex of a pair of spaces
- Prop: direct sum of chain complexes are also chain complexes.
- Prop: the homology group of X is the direct sum of homology group of its path components.
- Functoriality and Induced maps
- **Def: Category. Objects and morphisms**
- Def: composition rules for morphisms
- Ex: give some examples of morphisms
- **Def: functor**
- Def: chain maps
- Prop: {Chain complexes; chain maps} is a category
- **Thm: Homology defines a functor**
- Def: $f_\# : C_*(X) \to C_*(Y)$, where $f : X \to Y$ is a continuous map.
- Lem: $f_\#$ is a chain map
- Lem: The functorial property of $\square_\#$
- Rem: Here's a big picture of 2 functors

$$\begin{Bmatrix} \text{Spaces} \\ \text{Maps} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Chain CX} \\ \text{Chain maps} \end{Bmatrix} \longrightarrow \begin{Bmatrix} R-\text{modules} \\ R-\text{linear maps} \end{Bmatrix}$$

  Be comfortable with showing that each bracket is a category and each arrow is well defined and a functor.

Lecture 4

- Prop: Maps of pair functoriality. i.e. do the same chain of functors above, except with pair of spaces and map of pairs.
- <u>Homotopy invariance</u>
- Def: $g_0, g_1 : C \to C'$ are chain maps. They are chain homotopic if ...
- Lem: Chain homotopy is an equivalence relation.
- Def: Two chain complexes are chain homotopy equivalent if...
- Prop: If for two chain maps, $g_0 \sim g_1$ then $g_{0*} = g_{1*}$
- Cor: $C \sim C' \implies H_*(C) = H_*(C')$
- Thm: Universal Chain Homotopy. Section completely skipped.

Lecture 5

- Cor: Three corollaries of Universal Chain Homotopy:
  - If $f_0 \sim f_1$ then $f_{0*} = f_{1*}$
  - If $f : X \to Y, g : Y \to X$ gives homotopy equivalence then they are both isomorphisms
  - If X is contractible what is $H_*(X)$?
- Exact sequence, SES of chain complexes, SES of a pair
- **Thm: Snake lemma**
- Cor: LES of pair
- Lem: $\tilde{H}(X) \cong H(X, \{p\})$

Lecture 6

- Subdivision: $C_*^U(X)$
- Prop: $C_*^U(X)$ is a subcomplex
- **Thm:** $i_*(H_K^U(X) \to H_K(X)$ **is isomorphism**
- Defn: Mayer Vieroris sequence
- **Prop: MVS is a short exact sequence**
- Prop: Mayer viertoris long exact sequence
- **Prop:** $\tilde{H}_i(S^n)$
- Notatoin: $[S^n]$

Lecture 7

- Lemma: Turning a commmuting diagram of 2 SES into a commuting diagram of 2 LES.
- As a result of the lemma, you can turn two MVS SES into two MVS LES that commutes
- Defn: $r_n : S^n \to S^n$
- **Prop:** r_{n*} **maps** $[S^n]$ **to** $-[S^n]$

- Cor: r_{*v} also maps to $[S^n]$ to $-[S^n]$
- Def: Deformation retract, Good pair
- Thm: good pair isomorphism on homology, no proof needed
- Compute $S^2/\{N, S\}$. Compute $S^1 \times S^1/S^1 \times 1$. They are the same. But latter helps to compute $H_*(S^1 \times S^1)$

Lecture 8

- *Below helps prove Excision*
- Thm: Five Lemma
- Defn: $C_*^U(X, A)$
- Lemma: $H_*^U(X, A) \to H_*(X, A)$ is isomorphic
- **Thm: Excision**
- *Below helps prove collapsing of a pair*
- Prop: LES of a triple
- Lemma: Deformation retraction induces isomorphism on relative homology
- **Thm: Collapsing of a pair**
- Def: Manifold
- Thm: Relative homology of a manifold
- *Remember how conditions of Excision and collapsing of pair differ!*

2 Cellular Homology

Lecture 9

- Defn: Degree of a map $f : S^n \to S^n$
- **Properties of degree map**
  - Degree of antipodal map
  - Degree of reflection
  - homotopic equivalent same degree, homeomorphism then same degree
  - **Local degree Stuff**
- Concept: $\pi_* : H_n(S^n) \to H_n(S^n, S^n - p)$, is an isomorphism. Identify $\pi_*[S^n]$ with $[S^n, S^n - p]$. Then, use excision to denote $[U, U - p] \to [S^n, S^n - p]$
- Prop: $[U', U' - p] \to [U, U - p]$ is an isomorphism
- **Defn: Local degree of a map**
- Prop: Local degree does not depend on the choice of neighbourhood

- Prop: $V = \coprod U_i$. Then by excision, show that $j_* : H_n(V, V - f^{-1}(p)) \cong H_n(S^n, S^n - f^{-1}(p))$ hence $[u_i, u_i - p]$ is a genrator for $H_n(S^n, S^n - f^{-1}(P))$
- Prop: the structure $H_n(S^n) \to H_n(S^n, S^n - p)$.
- **Thm (big): Degree of f as sum of local degrees**

Lecture 10

- Def: Attaching via a function
- Def: attaching via k ccell
- **Defn: finite cell complex, k skeleton.**
- Cell structures of S^k, D^k, graphs
- Def: Wedge Product
- Prop: $\mathbb{CP}^n \cong S^{2n+1}/S^1$
- **Defn: Hopf map**
- **Prop: using Hopt map to construct $\mathbb{CP}^n$**
- **Thm: Cellular construction of $\mathbb{CP}^n$ and computing** $H_*(\mathbb{CP}^n)$ LES break up so we get direct sum.

Lecture 11

- Prop: $H_k(D^k, S^{k-1}) \cong H_{k-1}(S^{k-1})$
- Prop: (X_k, X_{k-1}) is a good pair
- Prop:
  - $X_k/X_{k-1} \simeq \bigvee S^k$
  - $H_k(X_k, X_{k-1})$, Generated by e_α
- Defn: ρ_β map: prjection onto the βth cell. ρ_β works like δ_{ij} for e_α
- **Def:** d_k^{cell}
- **Lemma:** $d_k = (\pi_{k-1})_* \circ \delta_k$
- **Cor:** $d_k \circ d_{k+1} = 0$
- **Defn:** $C_i^{cell}(X)$
- **Big Thm:**
  - $H_*^{cell}(X) = H_*(C_*^{cell}(X)) \cong H_*(X)$
  - A way to compute H_*^{cell}

Lecture 12

- Lemma: let X be a fcc, where it has one 0 cell. ALl the rest of the cells has dim d with $m \leq d \leq n$. Then $H_*(d) = 0$ for all $* < m$ or $* > n$.
- Lemma: X a FCC, then (X, X_k) is a good pair
- Cor: $H_k(X_{k+1}) = H_k(X)$: proof is by LES of pair plus collaping of a pair
- **Thm: X a fcc then** $H_*^{cell}(X) \cong H_*(X)$

2.1 Homology with coefficients

- Defn: Tensor product
- $\otimes M$ functor
- Defn: singular chain complex with coefficient in G
- Defn: Euler character
- Thm: $\chi(X) = \chi(H_\bullet(X))$
- Eilenberg Steenrod axioms

Lecture 13

- Def: Free resolution
- Def: $\text{Tor}_i(M, N)$ is well defined
- Fact: what is Tor_0?
- Def: short injective chain complex
- **Thm: Structure theorem for chain complex over PID**
- Cor: Two free chain CX over PID have $\cong$ homology then then are homotopic equiv. Proof skipped, doesn't seem to be tested
- Cor: If C is a chain CX over a field $\mathbb{F}$, then $C \cong (H_*(C), 0)$ Proof skipped, doesn't seem to be tested
- **Cor: The Universal coefficient theorem (UCT)** Proof skipped, doesn't seem to be tested

3 Cohomology and Products

- *If M, N are R modules then so is* $\hom(M, N)$
- Def: $f : M_1 \to M_2$, what is f^*?
- Def: Contravariant functor
- Prop: f^* is a contravariant functor
- Def: $(\hom(C, N), d^*)$ cochain complex
- Def: The contravariant functor (Chain complex) to (Cochain complex)
- **Def: Cohomology**
- Prop: Draw the functorial diagram for how to go from pair of spaces to cohomology of pairs

Lecture 14

- C_i^* explicitly. What are elements in C_i^* specified by?
- Prop: $(d^*)^2 = 0$, $d^*\alpha = \alpha d_*$
- Why $d^*(\alpha)(\sigma) = (\alpha \circ d_*)\sigma$
- Defn: Cochain maps: maps between cochain of different spaces. i.e. $f^\#(\alpha)(\sigma) = \alpha f_\# \sigma$

- Prop: $f^\#$ is a cochain map, that is $f^\# d^* = d^* f^\#$. I.e. commutes with boundary. Proof kind of messy.
- **Thm: $f^\#$ induces f^***
- Defn: Two cochains are cochain homotopic if...
- Lemmas: $f \sim g$ implies $f^* \sim g^*$.
- Lemma: If $f, g : C \to C'$ are chain hty, $f \sim g$ via h then...
- **Prop: Eilenberg- Steenrod axioms, hence give you four properties of cohomology** (1. homotopic $\implies$ same homologies. 2. LES of pair. 3. Excision 4. Dimension)
- Thm: Any functor satisfying above axioms is...
- Thm: Cohomologies and cellular cohomologies are iso if...

3.1 EXT and UCT

- Defn: $\mathrm{Ext}^i(M, N)$, $\mathrm{Tor}_i(M, N)$. What is Ext^0? what is Tor_0?
- Example: compute $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z})$
- **Thm: Write H_i, H^i in terms of Tor and Ext**. Using this, we get example of if we have fcc X, H_k, H^k can be written as direct sum of free and torsion components. Proof doesn't seem to be tested.

3.2 Pairing

- Def: Given C a chain complex over R, how to make a bilinear pairing
- **Thm: $H^k \times H_k$ descended from above.**

Lecture 15

3.3 Cup Products

- **Def: Cup Product**
- Lemma: $\cup$ Makes $C^k(X; R)$ into a commutative ring
- Lemma: Leibniz rule
- Cor: $\cup$ descends to map on $H^*(X; R)$
- **Prop: Continuous maps induce ring hom between cohomologies**. Note that this is true for H^k but NOT necessarily true for C^k.
- Prop: $\cup$ on H^* is graded commutative. Proof skipped, require the map that maps to mirror-identity in simplices

Lecture 16

- Thm: $r \sim 1_{C_*(X)}$ (This finishes the proof for graded commutativity)
- **Defn: pairs using** $\mathbb{Z}$ coefficients: $C^*(X, A)$
- Prop: what can you say about $C^*(X, A) \times C^*(X)$?
- **Cor** $\cup$ descends to a map $H^*(X, A) \times H^*(X) \to H^*(X, A)$

- Cor: Generally $\cup$ defines a map $H^*(X,A) \times H^*(X,B) \to H^*(X, A \cup B)$ Proof?
- Examples of cup products and cohomology
  - $H^0(X)$ when X is path connected
  - $H^*(S^n, G)$
  - Skipped
  - Structure of $H^*(X \coprod Y) \cong H^*(X) \oplus H^*(Y)$ as direct product of rings Proof omitted since this is better to be tested via using
  - $H^*((X,p) \vee (Y,q))$ two spaces attached at p, q.

3.4 Exterior products

Lecture 17

- **Defn: Exterior product**, defined using $\times$.
- Prop: properties about the exterior product
  - $H^*(X,A) \times H^*(Y) \to H^*(X \times Y, A \times Y), (a,b) \mapsto a \times b$
    extends to
    $H^*(X,A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y)$
  - Distribute $(a_1 \times b_1) \cup (a_2 \times b_2)$
  - **Thm: The exterior product isomorphism** The proof is quite long and technical.
    * $\overline{h}, \underline{h}$ as contravariant functors
  - Compute the following:
    * Ring structure of $H^*(T^2)$
    * (group structure of) $H^*(S^2 \times S^2)$
    * Deduce $S^2 \times S^2$ is not homotopic equiv to $S^2 \vee S^2 \vee S^4$ despite having same homologies.
    * $H^*(\Sigma_g)$
  - Convention for computing using exterior product or cup product:
    * Write $(a_1, \ldots, a_n)$ be generators H^i of some space
    * Then, write $a, b = (0, 0, \ldots, a_1)$, or something like that, etc.
    * Write the dimension chart
    * $(a_1 \times b_1) \cup (a_2 \times b_2)$

Lecture 18

- Focused on proving the isomorphism for exterior product

Lecture 19

4 Vector bundles

- **Defn: n-diml vector bundle**
- Complex vector bundle
- **Defn: Morphism**

- Defn: bundle isomorphism
- Defn: sub-bundle
- Defn: section, non-vanishing section, trivial bundle
- Example: trivial bundle
- Prop: equivalent condition of being a trivial bundle
- Example: mobius bundle
- **Example: tautological bundle and tangent sphere bundle** Unfamiliar with this one
- Def: pullbacks of vector bundles. What are local trivialisations in this case?
- Lemma: pullback of vb can be composed
- Def: restriction to a smaller base of vector bundles
- Lem: non-vanishing sections can be pulled back
- Example: $\mathbb{RP}^n$ is nontrivial
- **Defn: product of two vector bundles**
- **Defn: Whiteney sum**
- Defn: supp of a function
- Defn: Partition of unity subordinate to a cover, admits PoU
- **Thm: Big theorem:** $E\mid_{B\times 0}\simeq E\mid_{B\times 1}$
-

Lecture 20

- Set of lemmas to prove big theorem
  - If $E\mid_{B\times[0,1/2]}$ and $E\mid_{B\times[1/2,1]}$ are trivial, then is E
  - For each $b\in B$, there exists an open nbd U_b such that $E\mid_{U_b\times I}$ is trivial.
  - This proves the theorem
- Cor: $\pi: E\to B$ is a v.b. If $g_0, g_1: B'\to B$, and $g_0\sim g_1$ via $h: B'\times I\to B$, then $g_0^*\sim g_1^*$
- Cor: If B is contractible and admits PoU then it is trivial.

4.1 Riemannian metrics, Thom Iso, Euler class, etc

- **Riemannian Metric**
- **Defn: unit disk, unit sphere bundle**: note: they are not vector bundles. Along with map π restricted on there. $\pi: D_g(E)\to B, \pi: S_g(E)\to B$.
- Prop: the choice of Riemannian metrid doesn't matter
- Example: if two bundles are trivial, what is an R-metric? $S(B\times\mathbb{R}^n) - B\times S^{n-1}$ if it were trivial. This shows the real tautological and complex tautological bundles are nontrivial.

- Prop: Given a vector bundle, if B has a PoU then it has a R metric (proof: the riemannian metric is given by sum of R-metric at each of the member of the open cover of the local trivialisation)
- **Defn: the Thom class**
  - Maps s_0, i_b, spaces $E_b, E_b^{\#}, E^{\#}$
  - Definition of thom class U

Lecture 21

- Example: relating $H^*(B)$ to $H^*(E, E^{\#})$
- Construction: Let $\pi : E \to B$ be a bundle. Let $B' \to B$ be a map. There is a bundle isomorphism between E and $f^*(E)$
- Lemma: If U is a R Thom class for E then F^*U is a R thom class for $f^*(E)$.
- Lemma: $B = B_1 \cup B_2$. $U \in H^n(E, E^{\#})$. Then what condition makes U a TC for E?
- **Thm: the Thom Isomorphism Theorem**
- Note: $E^{\#} \sim S(E)$
- **Def: Gysin sequence**
- **Def: Euler class**
- **Remark: The LES you're supposed to remember**

Lecture 22

- **Defn: admits an orientation**
- **Thm: Properties of** e
  - e behaves naturally under pullback
  - If E is trivial and $n > 0$ then $e(E) = 0$
  - $e(E_1 \oplus E_2) = e(E_1) \oplus e(E_2)$
  - If E has a non-vanishing section then $e(E) = 0$

  Two things to remember: E is trivial $\iff$ it is the pullback of the trivial bundle over a point.

  If S is a nonvanishing section then $E = \langle s \rangle \oplus \langle s \rangle^{\perp}$
- **Thm: solving** $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$
- Cor: $\pi_3(S^2) \neq 0$

4.2 Manifolds

Lecture 23

- **Def: n-manifold,** smooth manifold, transition functions
- Defn: $(M \mid A)$. What is $H_*(M \mid x)$? What is $H^*(M \mid x; R)$?
- Defn: R fundamental class
- **Thm: if** $A \subset M$ **is compact then** $(M| \mid A)$ **admits a unique** $\mathbb{Z}/2$ **fundamental class**

- Defn: Orientable
- Defn: submanifold
- Defn: normal bundle
- **Thm: Tubular neighbourhood theorem**
- Prop: $E = E_1 \oplus E_2$. Comment on orientatbility.
- Thm: M orientable $\iff$ its normal bundle is

Lecture 24

- Below is on poincare duality
- Work in fields
- **Thm: Poincare duality**
- Defn: cap product, intersection pairing, algebraic PD, geometric PD

5 Formula Sheet

5.1 Possible $R-$modules of a topological space

- $C_*^U(X, A)$

5.2 Tools to compute homology groups

- Snake lemma / LES of a pair / LES of a triple
- Deformation retraction
- Excision
- Collapsing of a pair
- Universal coefficient theorem:
  $H^i(X; G) \cong Hom(H_i(X; G)) \oplus Ext^1(H_{k-1}(X); G)$
- Cohomology of wedge sums
- Cohomology of disjoint unions
- Isomorphism on exterior product

5.3 Some rules going from Ho to Co

- $d^*(\alpha)(\sigma) = \alpha d_*(\sigma)$
- $f^\bullet(\alpha)(\sigma) = \alpha f_\bullet(\sigma)$

5.4 Topological spaces

- T^k
- ΣX where $H_k(\Sigma X) = H_{k+1}(X)$
-

5.5 You should know

- Cellular decomposition of $\mathbb{CP}^n$, S^n
- Homologies:
  - S^n
  - T^n
  - Σ_n
  - $\mathbb{CP}^n$
  - $\mathbb{RP}^n$
  - $S^2/\{N, S\}$
  - $S^1 \times S^1 / S^1 \times 1$
  - Manifold $(M, M - x)$
- Cohomologies:

- $H^*(\Sigma_g)$
- $\mathbb{CP}^n$
- $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$.
- Attention: $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ vs $H^*(\mathbb{RP}^n; \mathbb{Z})$

- Ring structures of
  - $H^*(S^n)$, write it in terms of $\mathbb{Z}[\alpha]/\alpha^2$
  - $H^*(S^2 \vee S^2 \vee S^4)$
  - $H^*(T^2)$
  - (group structure of) $H^*(S^2 \times S^2)$
  - Σ_g

-