week 1 lec 1.

Convention I, I^n, S^n, D^n, $D^n/S^{n-1} \cong S^n$

def homotopic maps

let X, Y be spaces, $f_0, f_1 : X \to Y$ are homotopic if $\exists$ $F: X \times I \to Y$ s.t. $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ $\forall x \in X$

and F is a homotopy.

$f_t(x) = F(x,t)$, $f_t : X \to Y$. f_t is a path from f_0 to f_1 in $map(X,Y)$

example of homotopic maps

1) $1_{\mathbb{R}^n}$, $0_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ homotopic via $F(x,t) = x(1-t)$

2) $S^1 \to S^1$,

$A_1 : v \mapsto -v$ $\qquad A_1 \sim 1_{S^1}$ via $f_t(z) = e^{i\pi t} z$

$1_{S^1} : v \mapsto v$

lemma 1 homotopy is an $\sim$

lemma 2 if $f_0 \sim f_1 : X \to Y$ then $g_0 \circ f_0 \sim g_1 \circ f_1$ $X \to Z$.

$g_0 \sim g_1 : Y \to Z$

def $[X,Y] = maps(X,Y)/\sim$

Prop $[X, \mathbb{R}^n]$ has one element

let $f: X \to \mathbb{R}^n$

$f = 1_{\mathbb{R}^n} \circ f \sim 0_{\mathbb{R}^n} \circ f = 0$

defn contractible spaces

A space X is contractible if $1_X \sim c_p$ ↙ constant map that maps to a point.

Prop Y is contractible $\Leftrightarrow$ $[X,Y]$ has 1 elemt. $\forall$ space X.

$\Rightarrow$ let $f \in [X,Y]$. then $f = 1_X \circ f \sim c_p \circ f = c_p$.

$\Leftarrow$ $[Y,Y]$ has only one elemt so $\sim$ to c_p.

def Spaces X, Y are hom equiv if $\exists f: X \to Y,\ g: Y \to X,$

s.t $f \circ g = 1_Y$ $g \circ f = 1_X$.

ex $\mathbb{R}^n \sim \{0\}$, contracible space $\sim \{0\}$, $\mathbb{R}^n \setminus \{0\} \sim S^{n-1}$

def pairs of spaces

↳ Pair of spaces: (X, A), $A \subset X$.

↳ map of pairs: $f: (X, A) \to (Y, B)$ cts map, $f(A) \subset B$.

↳ maps of pairs $f_0, f_1: (X, A) \to (Y, B)$ is homotopic if $f_0, f_1: X \to Y$ are homotopic via $H: (X \times I, A \times I) \to (Y, B)$

↳ $f: (X, A) \to (Y, B)$
$g: (Y, B) \to (Z, C)$ $\Rightarrow$ $g \circ f: (X, A) \to (Z, C)$

def homotopy groups

If X is a space, $p \in X$, then the homotopy group

$$\pi_n(X, p) = [(I^n, \partial I^n), (X, p)]$$
$$= [(D^n, S^{n-1}), (X, p)]$$
$$= [(S^n, *), (X, p)].$$

Prop. Properties of homotopy groups

note: $\pi_0(X, p) = \{$path cmpts of X$)$

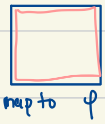
$\pi_1(X, p)$ is a group.

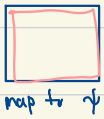
$\pi_n(X, p)$ is an abelian group. when $n > 1$.

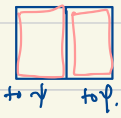
Prop. $\pi_n(X, p)$, is a group.

① addition $\varphi, \psi: (I^n, \partial I^n) \to (X, p)$

$\varphi + \psi: (I^n, \partial I^n) \to (X, p)$





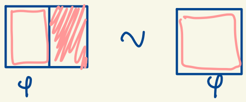


② Identity map

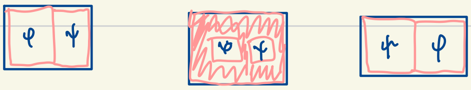
$e: (I^n, \partial I^n) \to (X, p)$

$x \mapsto p$

$[\varphi + e]$



③ abelian-ness for $n \geq 1$.



④ $\varphi^{-1} = \varphi \circ r$ where $r = I^n \to I^n$

$(t_1, \cdots t_n) \longmapsto (1 - t_1, \cdots t_n)$

$(\varphi + \varphi^{-1})(s_1, s_2, \cdots s_n)$

$$= \begin{cases} \varphi(2s_1, s_2, \cdots s_n) & s_1 \in [0, 1/2] \\ \varphi^{-1}(2s_1 - 1, s_2, \cdots s_n) & s_1 \in [1/2, 1] \end{cases}$$

$= \varphi(2-2s_1, s_2, \cdots s_n)$ "like undoing it"

functoriality

$$\begin{tikzcd} (I^n, \partial I^n) \arrow[d, "\varphi"'] \arrow[dr, "f \circ \varphi"] & \\ (X, p) \arrow[r, "f"] & (Y, q) \end{tikzcd}$$

$f : (X, p) \to (Y, q)$ induces $f_* : \pi_n(X, p) \to \pi_n(Y, q)$

$f_*([\varphi]) = [f \circ \varphi]$

(check) $\boxed{(f \circ g)_* = f_* \circ g_*}$

$f : (X, p) \to (Y, q)$ $\quad f_* : \pi_n(X, p) \to \pi_n(Y, q)$

$g : (Y, q) \to (Z, c)$ $\quad g_* : \pi_n(Y, q) \to \pi_n(Z, c)$

$f \circ g : (X, p) \to (Z, c)$

$g_* \circ f_*([\varphi])$

$= g_* \circ [f \circ \varphi]$

$= [g \circ f \circ \varphi]$

$= (g \circ f)_*([\varphi])$

homotopy invariance

If $f_0, f_1 : (X, p), (Y, q)$, and $f_0 \sim f_1$, then $f_{0*} = f_{1*}$

as $f_{0*}([\varphi]) = [f_0 \circ \varphi] = [f_1 \circ \varphi] = f_{1*}([\varphi])$

$\varphi : (I^n, \partial I^n) \to (X, p) \in \pi_n(X, p)$.

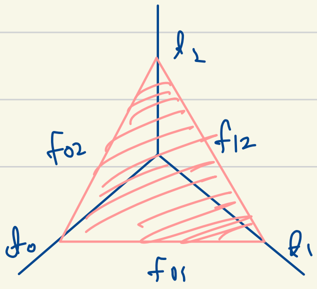
week 1 lecture 2

I) Singular homology.

Def the n-simplex $\Delta^n = \{(t_0, t_1, \cdots t_n) \in \mathbb{R}^n \mid t_i \geq 0, \Sigma_i t_i = 1\}$.

Def faces: if $I \subset \{0, 1, \cdots n\}$,

$f_I = \{\vec{t} \in \Delta^n \mid t_i = 0 \text{ if } i \notin I\}$



$f_{12} = \{\vec{t} \in \Delta^n, t_0 = 0\}$

f_{012} : whole thing

def face maps

If $I = \{i_0 < i_1 < \dots < i_k\} \subseteq \{0, \dots n\}$

$$F_I : \Delta^{|I|-1} \longrightarrow f_I \subset \Delta^n$$

$$F_I(\vec{t}) = \vec{x} \quad \text{where } x_i = \begin{cases} 0 & i \notin I \\ t_j & i = i_j \end{cases}$$

$F_{12} : \Delta^1 \to f_{12}$

$(t_0, t_1) \mapsto (t_0, t_1, 0)$

$F_{02} : \Delta^1 \to f_{02}$

$(t_0, t_1) \mapsto (t_0, 0, t_1)$

$$\boxed{F_I : \Delta^{|I|-1} \longrightarrow f_I \subset \Delta^n}$$

Δ^n is typically a n dim submanifold in $\mathbb{R}^{n+1}$

homeomorphism.

F_I is a map $\Delta^{\text{smaller}\cdot\text{dim}} \hookrightarrow \Delta^{\text{bigger}\cdot\text{dim}}$

def chain complex

R be a commutative ring. A chain cx over R (C, d) is

1) R modules C_i, $i \in \mathbb{Z}$ $\quad C = \bigoplus_{i \in \mathbb{Z}} C_i$

2) R-linear maps $d_i : C_i \to C_{i-1}$ $\quad d = \bigoplus_{i \in \mathbb{Z}} d_i$

$$\begin{cases} d : C \to C \\ d(C_i) \subset C_{i-1} \end{cases}$$

s.t.

3) $d_i \circ d_{i+1} : C_{i+1} \to C_i = 0 \quad \forall i.$

def. i^{th} homology group.

$d_i \circ d_{i+1} = 0 \Rightarrow \operatorname{im}(d_{i+1}) \subset \ker(d_i)$

the i^{th} hom group $H_i(C) = \dfrac{\ker(d_i)}{\operatorname{im}(d_{i+1})}$ it's an $R\cdot$module.

$H_*(C) = \bigoplus_i H_i$

def : $x \in \ker(d)$, x is closed / cycle

$x \in \operatorname{im}(d)$, x is exact / boundary.

If $dx = 0$, $[x]$ be its image in $H_*(C)$.

def. The chain complex of the n-simplex is $(S_*(\Delta^n), d)$

$S_k(\Delta^n)$ is the free module generated by k-dim'l faces.

$S_k(\Delta^n) = \langle f_I \mid |I| = k+1 \rangle$ $S_k(\Delta^n) = 0$ for $k < 0$.

$d(f_I) = \sum_{j=0}^{k} (-1)^j f_{I \setminus \{i_j\}}$

Prop $d^2 = 0$

enough to check $d^2(f_I) = 0$ since f_I is a basis.

index hiccup! Come back later.

Note: no face maps. just fixed Δ^n. get k-dim'l faces.

Example : chain complex of 2-simplex

$\langle f_0, f_1, f_2 \rangle / \langle f_0 - f_1, f_1 - f_2, \cancel{f_2 - f_0} \rangle$

$S_{-1}(\Delta^2) = \langle \ \rangle$

$S_0(\Delta^2) = \langle f_0, f_1, f_2 \rangle$

$S_1(\Delta^2) = \langle f_{01}, f_{02}, f_{12} \rangle$

$S_2(\Delta^2) = \langle f_{012} \rangle$

d_0: $\mathrm{Im}(d_0) = \phi$, $\ker(d_0) = \langle f_0, f_1, f_2 \rangle$

d_1: $\mathrm{Im}(d_1) = \sum a_i f_i$ $\sum a_i = 0$, $\ker(d_1) = \langle f_{01} + f_{20} + f_{12} \rangle$

d_2: $\mathrm{Im}(d_2) = \langle f_{12} \overset{\shortparallel}{-f_{02}} + f_{01} \rangle$, $\ker(d_2) = 0$

$H_0 = \dfrac{\ker(d_0)}{\mathrm{im}(d_1)} = \mathbb{Z}$

$H_1 = \dfrac{\ker(d_1)}{\mathrm{im}(d_2)} = 0$

$H_2 = \dfrac{\ker(d_2)}{\mathrm{im}(d_3)} = 0$

def reduced chain complex of Δ^n

$(\tilde{S}_*(\Delta^n), d)$ $\begin{cases} \text{if } k \neq -1 & \tilde{S}_k(\Delta^n) = S_k(\Delta^n) \\ \text{if } k = 1 & \tilde{S}_{-1}(\Delta^n) = \langle f_\phi \rangle \text{ if } |I| = 1,\ d f_I = f_\phi. \quad d f_\phi = \langle \phi \rangle. \end{cases}$

idea: want H_0 to be trivial.

$S_{-1}(\Delta^2) = \langle f_\phi \rangle$

$S_0(\Delta^2) = \langle f_0, f_1, f_2 \rangle$

$S_1(\Delta^2) = \langle f_{01}, f_{12}, f_{21} \rangle$

$\ker d_0 = 0$

$\mathrm{Im}\ d_1 = \sum a_i f_i,\ \sum_i = 0$

$\Rightarrow H_0 = \dfrac{\ker(d_0)}{\mathrm{im}(d_1)} = 0$

def singular chain complex

Let X be a top space. Then its singular chain complex is $(C_*(X), d)$ $C_k(X) = \{ \sigma : \Delta^k \to X .\ \text{continuous} \}$ is the free $\mathbb{Z}$-module generated by all $\sigma : \Delta^k \to X$.

def elements in $C_k(X)$ is written as $\sum a_i \sigma_i$ (finite $\sum$s.) $a_i \in \mathbb{Z}$ $\sigma_i : \Delta^k \to X$.

singular simplex : $\Delta^n \to X$

def differential of singular chain complex

suffices to define d on σ since σ are generators.

let $\sigma : \Delta^k \to X$, $d(\sigma) = \sum_{j=0}^{k} (-1)^j \sigma \circ F_{\hat{j}}$ $F_{\hat{j}} : \Delta^{k-1} \to \Delta^k$ is the face map.

Prop $d \circ \varphi_\sigma = \varphi_\sigma \circ d$

the map $\varphi_\sigma : S_*(\Delta^k) \to C_*(X)$

$f_I \mapsto \sigma \circ F_I$

$$\begin{tikzcd} S_k(\Delta^n) \arrow[r, "d"] \arrow[d, "\varphi"] & S_{k-1}(\Delta^n) \arrow[r, "d"] \arrow[d] & S_{k-2}(\Delta^n) \arrow[d] \\ C_k(X) \arrow[r, "d"] & C_{k-1}(X) \arrow[r, "d"] & C_{k-2}(X) \end{tikzcd}$$

satisfies $d \circ \varphi_\sigma = \varphi_\sigma \circ d$

Prop $d^2 = 0$ in $C_k(X)$

note $\sigma = \varphi_\sigma \circ F_{\{0,\dots,n\}}$ so $d^2\sigma = d^2 \varphi_\sigma \circ F_{\{0..n\}} = \varphi_\sigma \circ d^2 F_{\{0,..n\}} = 0$

def. Singular homology on X

$H_i(C_*(X))$

Prop. computing $H_*(\{\cdot\})$

each $C_k = \langle \sigma_k \rangle$ where $\sigma_k : \Delta^k \to \{\cdot\}$

$d(\sigma_k) = (-1)^n \sum_{j=0}^{n} \sigma_k \circ F_{\hat{j}}$ note $\sigma_k \circ F_{\hat{j}}$ is the map sendy Δ^{k-1} to $\{\cdot\}$ so σ_{k-1}

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{k-1} & \text{if } n \text{ even} \end{cases}$$

so $\text{Im}(d) = \langle \sigma_1, \sigma_3, \sigma_5, \dots \rangle$

$\ker(d) = \{ \sigma_0, \sigma_1, \sigma_3, \sigma_5, \dots \}$

$\ker(d)/\text{Im}(d) = \langle \sigma_0 \rangle \Rightarrow H_i(\{\cdot\}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{o.w.} \end{cases}$

week 1 lecture 3

def reduced singular chain cx

$$\tilde{C}_k(X) = \begin{cases} C_k(X) & k \neq -1 \\ \langle \sigma_\phi \rangle & k = -1 \end{cases}$$

$d\sigma = \sigma_\phi$ if $\sigma \in C_0(X) : \Delta^0 \to X$

$d\sigma_\phi = 0$

makes $\ker(d_0) = \phi$ so makes $\tilde{H}_0(X) = 0$.

Prop: If X is path connected then $H_0(X) = \mathbb{Z} = \langle \sigma_p \rangle$ $\sigma_p : \Delta^n \to p \in X$.

Proof: $H_0(X) = \frac{\ker(d_0)}{\operatorname{im}(d_1)}$

$d_0 : C_0(X) \to C_{-1}(X)$ but $C_{-1}(X) = 0$ so $\ker(d_0) = C_0(X) =$ maps $\langle \sigma : \Delta^0 \to X \rangle$

$d_1 : C_1(X) \to C_0(X)$ let $\sigma : \Delta^1 \to X \in C_1(X)$ then $d\sigma = \sum_{i=0}^{n} \sigma \circ F_{\hat{i}} \, (-1)^i$

so $\operatorname{im}(d_1) = \operatorname{span}\{ d\sigma \mid \sigma : I \to X \}$

$= \operatorname{span}\{ \sigma_p - \sigma_{p'} \mid p, p'$ are path connected $\}$.

$= \operatorname{Span}\{ \sigma_p - \sigma_{p'} \mid p, p' \in X \}$.

$H_0(X) = \frac{\ker(d_0)}{\operatorname{im}(d_1)} = \frac{\langle \sigma : \Delta^0 \to p \in X \rangle}{\langle \sigma_p - \sigma_{p'} \rangle}$ $\Rightarrow$ only one equivalence class of maps.

so $\cong \mathbb{Z}$.

def subcomplex

If (X,d) is a chain complex over R, a subcomplex of (C,d) is

1) $A_i \subset C_i$ as submodules

2) $d(A_i) \subset A_{i-1}$

Prop. Properties of subcomplex

If (A,d) is a subcomplex of (C,d) then

1) (A,d) is a chain complex.

2) $(C/A, d)$ is a chain complex

$C/A = \bigoplus_i C_i / A_i$

$d_i(A_i) \subset A_{i-1}$ so d_i descends: $d_i : C_i/A_i \to C_{i-1}/A_{i-1}$

def. quotient complex

(X,d) defined as above, $(C/A, d)$ is the quotient complex.

Prop. $A \subset X \Rightarrow C_*(A)$ is subcomplex of $C_*(X)$

let $\sigma : \Delta^n \to A$ then $d\sigma = \sum_i (-1)^i \sigma \circ F_{\hat{i}}$ with image in A.

def. Singular chain complex of a pair of spaces.

let (X,A) be pair of space. then singular chain cx is

$C_*(X,A) = C_*(X) / C_*(A)$

Prop. direct sums of chain complexes are also cx.

let (C_α, d_α) be chain complexes. Then so is $(\bigoplus_\alpha C_\alpha, \bigoplus_\alpha d_\alpha)$

$H(\bigoplus_\alpha C_\alpha) \cong \bigoplus_\alpha H(C_\alpha)$

~~Prop. Homology group in terms of path components.~~

$H_k(X) = \bigoplus_\alpha H(X_\alpha)$ where X_α are path components of X.

Proof. let $\sigma \in C_k(X)$, $\sigma : \Delta^k \to X$ since Δ^k is connected,

$$map(\Delta^k, X) = \coprod_\alpha map(\Delta^k, X_\alpha)$$

$$C_k(X) = \bigoplus_\alpha C_k(X_\alpha) \Rightarrow H_k(X) = \bigoplus_\alpha H_k(X_\alpha)$$

Functoriality & induced maps

defn (Category)

a category is

1) a collection of objects.

2) for each pair of objects, A. B, a set of morphisms $f : A \to B$

with composition rule: $f : A \to B$, $g : B \to C$, $g \circ f : A \to C$.

s.t. 1) $h \circ (g \circ f) = (h \circ g) \circ f$

2) for each object, $1_A : A \to A$, $1_B : B \to B$ s.t. $f : A \to B$, $f = f \circ 1_A = 1_B \circ f$

$\left\{ \begin{matrix} \text{objects} \\ \text{morphisms.} \end{matrix} \right.$

examples. $\left\{ \begin{matrix} \text{R-modules} \\ \text{R-lin.maps} \end{matrix} \right\}$ $\left\{ \begin{matrix} \text{spaces} \\ \text{cts maps} \end{matrix} \right\}$ $\left\{ \begin{matrix} \text{pairs of spaces} \\ \text{maps of pairs} \end{matrix} \right\}$

defn functor

let $\mathcal{C}_1, \mathcal{C}_2$ be categories, functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ assigns

1) $A \in Obj(\mathcal{C}_1)$ to $F(A) \in Obj(\mathcal{C}_2)$

2) $f \in mor(\mathcal{C}_1)$ to morphism of $\mathcal{C}_2$, $f : A \to B$ to $F(f) : F(A) \to F(B)$

s.t. 1) $F(f \circ g) = F(f) \circ F(g)$

2) $F(1_A) = 1_{F(A)}$

def chain maps

let (C,d), (C',d') be chain cxs over R, chain maps $f:(C,d)\to(C',d')$

is 1) R-linear maps $f_i : C_i \to C'_i$ $f=\bigoplus_i f_i$, $f: C\to C'$, $f(C_i)\subset C_{i-1}$

2) $d'f=fd$

$$\begin{tikzcd} C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\ C_i' \arrow[r, "d_i'"'] & C_{i-1}' \end{tikzcd}$$

$d_i\circ f_{i-1}=f_i\circ d_i'$

Prop

$\left\{\begin{array}{l}\text{chain complexes}\\ \text{chain maps}\end{array}\right\}$ is a category

Proof: 1) $1_{(C,d)}: (C,d)\to(C,d)$ is a chain map

2) If $f:(C,d)\to(C',d')$ $g:(C',d')\to(C'',d'')$ are chain maps, so is $g\circ f$.

$$\begin{tikzcd} C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \\ C_i' \arrow[r, "d_i'"'] \arrow[d, "g_i"'] & C_{i-1}' \arrow[d, "g_{i-1}"] \\ C_i'' \arrow[r, "d_i''"'] & C_{i-1}'' \end{tikzcd}$$

thm. Homology defines a functor.

$\left\{\begin{array}{l}\text{chain cx over } \mathbb{Z}\\ \text{chain maps}\end{array}\right\} \xrightarrow{H_*} \left\{\begin{array}{l}R\text{-modules}\\ R\text{-liner maps}\end{array}\right\}$

There's 2 steps

1) $f:(C,d)\to(C',d')$ descends to $f_*: H_*(C)\to H_*(C')$

If $x\in\ker(d)$, $dx=0$, $f(dx)=d'f(x)=0$ so $f(x)\in\ker(d')$

If $x\in\operatorname{im}(d)$, $x=dy$. $f(x)=f(dy)=d'f(y)$ so $f(x)\subset\operatorname{im}(d')$

so $f:\ker(d)\to\ker d')$

descends to $f_*: \ker(d)/\operatorname{im}(d) \to \ker(d')/\operatorname{im}(d'))$

$$f_*: H_*(C)\to H_*(C') \qquad f_*([x])=[f(x)]$$

2) Functoriality.

$$C\longmapsto H_*(C)$$

and if $1_C:(C,d)\to(C,d)$ then $(1_C)_* = 1_{H_*(C)}$

if $\left.\begin{array}{l} f:(C,d)\to(C',d') \\ g:(C',d')\to(C'',d'')\end{array}\right\}$ then $(g\circ f)_* = g_*\circ f_*$

Proof: remember: $\mathrm{Id}_*(x) = [x]$ $\mathrm{Id}_*: C \to H(C)$

1) $(1_C)_* = 1_{H^*(C)}$

$f: C \to C$ $f_*: H^*(C) \to H^*(C)$

if $x \in C$, $(1_C(x))_* = (x)_* = [x]$

2) $(g \circ f)_* = g_* \circ f_*$

$f: C \to C'$ $f_*: H(C) \to H(C')$

$g: C' \to C''$ $g_*: H(C') \to H(C'')$

$$(g \circ f)_*(x) = [g \circ f(x)] = g_*[f(x)] = g_* \circ f_* \circ [x]$$

$$\left\{ \begin{matrix} \text{chain complex over } R \\ \text{chain maps} \end{matrix} \right\} \xrightarrow{H_*} \left\{ \begin{matrix} R\text{-modules} \\ R\text{-linear maps} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} (C, d) \\ f: C \to C' \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} H_*(C) \\ f_*: H_*(C) \to H_*(C') \end{matrix} \right\}$$

NOTE: at this point, $\{\text{chain cxs}\} \to \{R\text{-modules}\}$ is abstract.

there's no topological spaces at all.

Def given $f: X \to Y$, what $f_\#$?

let $f: X \to Y$ be cts maps,

define $f_\# : C_*(X) \to C_*(Y)$

$\sigma \mapsto f \circ \sigma \in \mathrm{map}(\Delta^k, Y)$

lemma $f_\#$ is a chain map

Proof $\sigma \in C_k(X)$ then

$$f_\# \circ d(\sigma) = f_\# \circ \left(\sum_{i=0}^{n} (-1)^i \sigma \circ F_{\hat{j}} \right) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ F_{\hat{j}}$$

$$d \circ f_\#(\sigma) = d(f \circ \sigma) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ F_{\hat{j}}$$

lemma functorial property of $\#$

$$\left(\begin{matrix} \text{spaces} \\ \text{cts maps} \end{matrix} \right) \longrightarrow \left(\begin{matrix} \text{chain cxs} \\ \text{chain maps} \end{matrix} \right)$$

WTS: $(I_X)_\# = Id_{C_*(X)}$ and $(g \circ f)_\# = g_\# \circ f_\#$

1) $(I_X)_\# = Id_{C_*(X)}$

let $\sigma \in C_*(X)$ then, $I_X : X \to X$

$$(I_X)_\# : C(X) \to C(X)$$
$$\sigma \mapsto \sigma.$$

so $(I_X)_\#(\sigma) = \sigma$ so $(I_X)_\# = Id_{C_*(X)}$

2) $(g \circ f)_\#(\sigma) = g \circ f \circ \sigma = g_\#(f \circ \sigma) = g_\# \circ f_\# \circ \sigma$

So this gives a functor

$$\left\{ \begin{matrix} \text{spaces} \\ \text{cts maps} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{chain cxs} \\ \text{chain maps} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} X & \longmapsto & C_*(X) \\ f : X \to Y & \longmapsto & f_\# : C_*(X) \to C_*(Y) \end{matrix} \right.$$

Remark: Big picture of 2 functors.

$$\left\{ \begin{matrix} \text{spaces} \\ \text{maps} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{chain cx} \\ \text{chain map} \end{matrix} \right\} \to \left\{ \begin{matrix} R\text{- modules} \\ R\text{-linr maps} \end{matrix} \right\}$$

$$\begin{matrix} X & C_*(X) & H_*(X) \\ f : X \to Y & f_\# : C_*(X) \to C_*(Y) & f_* : H_*(X) \to H_*(Y) \end{matrix}$$

need to show

1. # gives a chain map
2. functoriality

1. functoriality
2. $f_\#$ descends to f_*

is a category

week 2 lec 1

Prop Maps of pairs functoriality

let $f:(X,A)\to(Y,B)$ be maps of pairs. Then, $f_*:X\to Y$, have

if $\sigma\in C_*(X)$, $\sigma:\Delta^k\to A$ have $f_\#(\sigma)\subset B$ $\Rightarrow f_\#(C_*(A))\subset C_*(B)$.

$f_\#$ descends to a map

$$f_\#: C_*(X,A)\longrightarrow C_*(Y,B)$$

get functoriality between categories

$$\left\{\begin{matrix}\text{Pairs of space}\\ \text{map of pairs}\end{matrix}\right\}\longrightarrow\left\{\begin{matrix}\text{chain cxs of } \mathbb{Z}\\ \text{chain maps}\end{matrix}\right\}\longrightarrow\left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \mathbb{Z}\text{-linear mps}\end{matrix}\right\}$$

$$(X,A)\longmapsto C_*(X,A)\longmapsto H_*(X,A)$$

$$f:(X,A)\to(Y,B)\longmapsto f_\#:C_*(X,A)\to C_*(Y,B)\longmapsto f_*:H_*(X,A)\to H_*(Y,B).$$

Homotopy invariance

def. chain homotopic

let $g_0, g_1: C\to C'$ be chain maps.

then, g_0 is homotopic to g_1 ($g_0\sim g_1$) if there exist R- linear maps $h_i: C_i\to C'_{i+1}$

s.t. $d'h+hd=g_1-g_0$.

$$\begin{tikzcd} C_{i+1} \arrow[r] & C_i \arrow[r, "d"] \arrow[dl, "h"'] & C_{i-1} \arrow[dl, "h"] \\ C_{i+1}' \arrow[r, "d'"'] & C_i' \arrow[r] & C_{i-1}' \end{tikzcd}$$

$$\boxed{hd+d'h=g_1-g_0}$$

lem: chain homotopy is an ~

def chain homotopic equivalent

Chain complexes C, C' are chain hty equivalent if ∃ chain mps

$f:C\to C'$, $g:C'\to C$, s.t. $f\circ g\underset{\uparrow}{\sim}1_{C'}$, $g\circ f\sim 1_C$.

chain maps homotopic

lemma: chain hty is an ~

Prop if $g_0, g_1 : C \to C'$ are chain maps, $g_0 \sim g_1$

$$g_{0*} = g_{1*} : H_*(C) \to H_*(C')$$

Proof note: $g_{0*} : H_*(C) \to H_*(C')$

let $[x] \in H_*(C)$ then $dx = 0$

$$g_{0*}([x]) - g_{1*}([x]) = [g_0(x) - g_1(x)]$$
$$= [(g_0 - g_1)(x)]$$
$$= [(hd + d'h)(x)]$$
$$= \underset{d(x)=0}{[hd(x)]} + \underset{\overset{\shortparallel}{0}}{[d'h(x)]} = 0 \in C'_*(X).$$

Cor: $C \sim C' \Rightarrow H_*(C) \cong H_*(C')$

Proof: if $C \sim C'$, $\exists f : C \to C'$, $g : C' \to C$, $f \circ g \sim Id_{C'}$ $g \circ f \sim Id_C$

$f_* \circ g_* : H_*(C) \to H_*(C)$

$H_*(C) \to H_*(C') \to H_*(C)$

$\Rightarrow$ iso.

Idea If $f_0 \sim f_1 : X \to Y$ via $H : X \times I \to Y$, $\boxed{\text{Idea for incoming}}$

If $\sigma : \Delta^k \to X$, $g_0(\sigma) = f_{0*}(\sigma)$ $g_i : C_*(X) \to C_*(Y)$

$g_1(\sigma) = f_{1*}(\sigma)$

get $h(\sigma) = H(\sigma \times I)$

→ hiccup : graphical arg don't quite get DONT GET THIS

universal chain homotopy

def $\varphi_{c_0}, \varphi_{c_1}$ as maps. $\Delta^n \to \Delta^k$ $\Delta^n \to X$

let $\sigma : \Delta^k \to X$, then there is chain map $\varphi_\sigma : S_*(\Delta^k) \to C_*(X)$

given by $f_I \mapsto \sigma \circ F_I$.

$c_0, c_1 : \Delta^n \to \Delta^n \times I$

$c_0 : x \mapsto (x, 0)$

$c_1 : x \mapsto (x, 1)$

get $\varphi_{c_0}, \varphi_{c_1} : S_*(\Delta^n) \to C_*(\Delta^n \times I)$

unfamiliar w/ proof Details

Idea of universal chain homotopy:

Idea: triangulate $(\Delta^n \times I)$ & use chain homotopic maps

↳ universal chain homotopy

$$U_n : S_*(\Delta^n) \to C_{*+1}(\Delta^n \times I)$$

↳ $dU_n + U_n d = \varphi_{c_1} - \varphi_{c_2}$ is chain homotopy.

↳ proof is index magic.

↳ Digream of $S_*(\Delta^n)$

week 2 lec 2

As corollary of universal chain homotopy:

cor. $f_0, f_1 : X \to Y$, $f_0 \sim f_1$, then $f_{0*} = f_{1*}$

cor. if $f : X \to Y$, $g : Y \to X$ induces homotopy equivalence,

$$\begin{cases} f_* : H(X) \to H(Y) \text{ is an } \cong. \\ g_* : H(Y) \to H(X) \text{ is an } \cong. \end{cases}$$

cor. if X is contractible $H_*(X) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{o.w.} \end{cases}$

come back to universal chain hty later.

1.4) Subdivisions

def: exact sequence, exact at a module,

Remark: a sequence is exact ⇔ it's a chain complex with $H_i = 0 \ \forall i$

Examples)

1) $0 \to A \to 0$ exact $\Rightarrow$ $A=0$

2) $0 \to A \xrightarrow{f} B \to 0$ exact $\Rightarrow$ f is iso

3) $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is SES,

$i: A \to B$ injective $\pi: B \to C$ surjective

$B/\mathrm{Im}(i) = B/\mathrm{Ker}(\pi) \cong \mathrm{Im}(\pi) = C$

$B/i(A) \cong C$

defn SES of chain cxs

let A, B, C be chain cxs.

then $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is a SES of cx if

i) i, π are chain maps

ii) each i, $0 \to A_i \to B_i \to C_i \to 0$ is SES $\forall i$.

$$\begin{tikzcd}
 & \vdots \arrow[d, no head] & \vdots \arrow[d, no head] & \vdots \arrow[d, no head] & \\
0 \arrow[r] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & 0 \\
0 \arrow[r] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & 0 \\
0 \arrow[r] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & \bullet \arrow[r] \arrow[d, no head] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Eg of SES

Consider $A \subset X$. consider $C_*(A)$, $C_*(X)$, $C_*(X, A)$

$0 \to C_*(A) \xrightarrow{i} C_*(X) \xrightarrow{\pi} C_*(X)/C_*(A)$.

thm: Snake lemma

let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a SES of chain cxs.

then there is a LES on homology

$$\begin{tikzcd}
H_i(A) \arrow[r, "i_*"] & H_i(B) \arrow[r, "\pi_*"] & H_i(C) \arrow[dll, bend left, "\partial"] \\
H_{i-1}(A) \arrow[r, "i_*"] & H_{i-1}(B) &
\end{tikzcd}$$

∂ is the boundary map on LES.

Proof scheme

- define ∂, $\partial[c]=[a]$
- show exactness at $H_*(A)$, $H_*(B)$, $H_*(C)$

Proof ①

$$\begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\ 0 \arrow[r] & A_i \arrow[r, "i"] \arrow[d, "d"] & B_i \arrow[r, "\pi"] \arrow[d, "d"'] & C_i \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "i"'] \arrow[d] & B_{i-1} \arrow[r, "\pi"'] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\ & {} & {} & {} & \end{tikzcd}$$

note that it's commutative

define $\partial: H_i(C) \to H_{i-1}(A)$: let $[c] \in H_i(C)$ - have $dc=0$. π surjective, so $\exists b \in B_i$, $\pi(b)=c$.

$db \in B_{i-1}$ $\pi(db) = d\pi(b) = d(c) = 0$ $\Rightarrow$ $db \in \ker \pi = \operatorname{im} i \Rightarrow \exists a \in A_{i-1}$, $i(a)=db$

Claim $da=0$ so $[a] \in H_{i-1}(A)$. indeed $i(da) = di(a) = ddb = 0$ $\Rightarrow$ $da \in \ker i$, $\Rightarrow$ $da=0$

② show exactness.

$$\begin{tikzcd} H_i(A) \arrow[r, "i_*"] & H_i(B) \arrow[r, "\pi_*"] & H_i(C) \arrow[dll, "\partial"'] \\ H_{i-1}(A) \arrow[r, "i_*"] & H_{i-1}(B) & \end{tikzcd}$$

with a further ∂ arrow drawn into $H_i(A)$.

↳ exactness at $H_i(A)$: WTS $\ker(i_*) = \operatorname{im}(\partial)$

let $[a] \in \ker(i_*)$ so $i_*[a]=0 \in H_i(B)$

$\Leftrightarrow$ $i(a) = db$ for some $b \in B_i$ as $(i(a))_v=0$ so $i(a) = db$. for $B \in B_i$

$\Leftrightarrow$ $[a] = \partial[c]$, where $c = \pi(b)$ by construction of ∂.

$\Leftrightarrow$ $[a] \in \operatorname{im} \partial$.

↳ exactness at $H_i(B)$ WTS $\ker(\pi_*) = \operatorname{im}(i_*)$

let $[b] \in \operatorname{im}(i_*)$ so $\exists a \in A_i$, $i(a)=b$ exactness: $\pi(b)=0 \in C$

So $\pi_* b = 0 \in H_{i+1} C$ $[b] \in \ker(\pi_*)$

↳ exactness at $H_i(C)$ WTS $\ker(\partial) = \operatorname{im}(\pi_*)$

let $[c] \in \operatorname{im}(\pi_*)$ so $\pi_*([b]) = [c]$ for some $b \in B_i$

$\pi b = c$ for some b $\Rightarrow \exists a$, $i(a) = db$.

$[c] = \partial[a]$ as construction above.

<u>corollary</u> <u>Apply snake lemma to pair of spaces</u>

$$\begin{tikzcd} & & H_{i+1}(X,A) \arrow[dll, "\partial"'] \\ H_i(A) \arrow[r, "i_*"] & H_i(X) \arrow[r, "\pi_*"] & H_i(X,A) \arrow[dll, "\partial"] \\ {} & & \end{tikzcd}$$

Prop Write $H_i(X,p)$ in terms of $H_i(X)$

LES of $(X,\{p\})$ is:

$$\begin{tikzcd}
\overset{=0}{H_{i+1}(\{p\})} \arrow[r] & H_{i+1}(X) \arrow[r, "\cong"] & H_{i+1}(X,\{p\}) \arrow[dll] \\
\overset{=0}{H_i(\{p\})} \arrow[r] & H_i(X) \arrow[r] & H_{i+1}(X,\{p\}) \\
 & \vdots & \\
\overset{=0}{H_1(\{p\})} \arrow[r] & H_1(X) \arrow[r] & H_1(X,\{p\}) \arrow[dll, "\partial=0"'] \\
\underset{=\mathbb{Z}}{H_0(\{p\})} \arrow[r, "i_*"] & H_0(X) \arrow[r] & H_0(X,\{p\})
\end{tikzcd}$$

let $H_i(\{p\}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z}, & i=0, \text{gen. by } [\sigma_p] \end{cases}$

$H_{i+1}(X) \cong H_{i+1}(X,\{p\})$

note that $i_*([\sigma_p]) = [\sigma_p] \neq 0$ in $H_0(X)$

So injective so map $\partial = 0$

Why injective ?

So $i>0$, $\begin{cases} H_i(X) \cong H_i(X,\{p\}) \\ H_0(X) \cong H_0(X,\{p\}) \oplus \mathbb{Z} \end{cases}$

$$0 \xrightarrow{\partial=0} \underset{A}{\mathbb{Z}} \to \underset{B}{H_0(X)} \to \underset{C}{H_0(X,\{p\})} \to 0$$

$C \cong B/A$, $B \cong A \oplus C$.

Week 2 lec 3

lemma $\tilde{H}(X) \cong H(X,\{p\})$

Recall that $H_*(X) = \begin{cases} H_*(X,\{p\}) & * \neq 0 \\ H_0(X,\{p\}) \oplus \mathbb{Z} & * = 0 \end{cases}$

But we Claim $\tilde{H}_*(X) = H_*(X,\{p\})$ $\forall *$.

Proof: define $\tilde{C}_i(X,p) = \tilde{C}_*(X) / \tilde{C}_*(p)$

$\cong C_*(X) / C_*(p)$

$= C_*(X,p)$

$\Rightarrow \tilde{H}_*(X,p) = H_*(X,p)$

get SES $0 \to \tilde{C}_*(p) \to \hat{C}_*(X) \to \tilde{C}_*(X,p) \to 0$

Snake lemma gives

$$\begin{tikzcd}
\tilde{H}_*(p) \arrow[r] \arrow[d, equal] & \tilde{H}_*(X) \arrow[r] & \tilde{H}_*(X,p) \arrow[r] & \tilde{H}_{*-1}(p) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

this is 0 $\forall *$ b/c reduced homology

$\hat{H}_*(X) \cong \tilde{H}_*(X,p)$

So $\tilde{H}_*(X) \cong \tilde{H}_*(X,p) \cong H_*(X,p)$

Subdivision (mainly used to show MVS)

defn $C_k^{U}(X)$

let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X.

$C_k^{U} = \{\sigma : \Delta^k \to X$ s.t. $\sigma(\Delta^k) \subset U_\alpha$ for some $\alpha\}$.

Prop. $(C_k^{U}(X), d)$ is a subcomplex

let $\sigma \in C_k^{U}(X)$ so $\operatorname{im}(\sigma) \subset U_\alpha$ for some α. Then,

$d\sigma = \sum_{i=0}^{k} \sigma \circ F_{\hat{i}}$ also has image in U_α. So it's a subcomplex.

thm Subdivision lemma

let $i : C_k^{U}(X) \hookrightarrow C_k(X)$ be inclusion

thm states that if U is an open cover for X, we have

$i_* : H_k^{U}(X) \to H_k(X)$ is an isomorphism

Proof is skipped in class 😛

→ maybe take a look at proof idea?

def. Mayer vietoris sequence - commutative diagram of inclusions

Suppose $U_1, U_2 \subset X$, and $X \subset U_1 \cup U_2$, get $\{U_1, U_2\}$ is an open cover of X.

Have commutative diagram of inclusions

$$\begin{tikzcd}
 & U_1 \arrow[dr, "j_1"] & \\
U_1 \cap U_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & U_1 \cup U_2 \\
 & U_2 \arrow[ur, "j_2"'] &
\end{tikzcd}$$

here $j_1 \circ i_1 = j_2 \circ i_2$

Prop The MVS SES

the below chain complex is a SES

$$\begin{tikzcd}
 & ① & & ② & & ③ & \\
0 \arrow[r] & C_*(U_1 \cap U_2) \arrow[rr, "i"] & & C_*(U_1) \oplus C_*(U_2) \arrow[rr, "j"] & & C_*(U_1 \cup U_2) \arrow[r] & 0
\end{tikzcd}$$

where $i = \begin{bmatrix} i_{1\#} \\ i_{2\#} \end{bmatrix}$ $\quad j = [j_{1\#} - j_{2\#}]$

WTS exact at ①, ②, ③

① show i is injective note that both $i_{1\#}: C_*(U_1 \cap U_2) \subseteq C_*(U_1)$ are injective

$i_{2\#}: C_*(U_2 \cap U_2) \subseteq C_*(U_2)$

so $\begin{bmatrix} i_{1\#} \\ i_{2\#} \end{bmatrix}$ is injective

② show that j is surjective

let $c \in C_*^{U}(U_1 \cup U_2)$ (isomorphic to $C^{U}(U_1 \cup U_2)$

so $c = \sum_i a_i \sigma_i + \sum_j b_j \tau_j$ where $\operatorname{im} \sigma_i \subset U_1$, $\operatorname{im} \tau_j \subset U_2$

write $a = \sum_i a_i \sigma_i$, $b = \sum_j b_j \tau_j$

then, $(a, -b) \in C_*(U_1) \oplus C_*(U_2)$ is the elemet s.t $j(a, -b) = a+b = c$.

so j is surjective.

③ show that at ② is exact.

WTS $\operatorname{im}(i) = \ker(j)$

$\operatorname{im}(i) \subset \ker(j)$ because that the diagram commutes.

$\operatorname{im}(i) \supset \ker(j)$ let $(a,b) \in \ker(j)$ write $\begin{cases} a = \sum_i a_i \sigma_i, & \operatorname{im}(\sigma_i) \subset U_1 \\ b = \sum_j b_j \tau_j, & \operatorname{im}(\tau_j) \subset U_2 \end{cases}$

$\hookrightarrow$ the a_is, b_js, σ_is, τ_js p.w. distinct

so $a \in C_*(U_1)$ $b \in C_*(U_2)$

$j(a,b) = 0 \Leftrightarrow j_{1\#}(a) = j_{2\#}(b)$ so $\sum a_i \sigma_i = \sum b_j \tau_j$ so rearranging implies

that $a_i = b_j$, $\sigma_i = \tau_j$ $\operatorname{im}(a) \subset U_1 \cap U_2$, $\operatorname{im}(b) \subset U_1 \cap U_2$.

$\Rightarrow c = \sum a_i \sigma_i \in C_*(U_1 \cap U_2)$, so $(a,b) \in \operatorname{im}(i)$ so $\ker j = \operatorname{im} i$.

<u>Cor. MVS SES + Snake lemma</u> $\rightarrow$ MVS

$U_1, U_2 \subset X$ and $U_1 \cup U_2 = X$, there's a LES

$$H_i(U_1 \cap U_2) \xrightarrow{i} H_i(U_1) \oplus H_i(U_2) \xrightarrow{j} H_i(U_1 \cup U_2) \xrightarrow{\partial} H_{i-1}(U_1 \cap U_2)$$

$$\text{IS}$$

$$H_i^{U}(U_1 \cup U_2)$$

<u>Cor MVS LES also work for $\tilde{c}$</u>

<u>Prop $\tilde{H}_i(S^n)$</u> $= \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$

Proof : By induction on n

Base case: $n = 0$

$S^0 = \{-1, +1\}$. $H_*(S^0) = H_*(\{-1\}) \oplus H_*(\{+1\})$

$= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{if } * \neq 0 \end{cases}$ $\Rightarrow \tilde{H}_0(S^0) = \mathbb{Z}$

$\uparrow$

Recall $H_*(X) = \begin{cases} H_*(X,p) & i \neq 0 \quad = \tilde{H}_*(X) \\ H_0(X,p) \oplus \mathbb{Z} & i=0 \quad = \tilde{H}_0(X) + \mathbb{Z} \end{cases}$

inductive step : $n>0$

compute $\tilde{H}_i(S_n)$

Write $S_n = U_+ \cup U_-$ $\quad U_+ = S^n \setminus \{(1,0,\cdots 0)\} \cong D^n$

$U_- = S^n \setminus \{(-1,0,\ldots 0)\} \cong D^n$

$U_+ \cap U_- = S^n \setminus \{(1,0,\ldots 0), (-1,0,\ldots 0)\} \cong I \times S^{n-1} \sim S^{n-1}$

By MV:

$$\begin{tikzcd}
{} \arrow[r] & \overset{\tilde{H}_{i+1}(S^{n-1})}{\tilde{H}_{i+1}(U_+ \cap U_-)} \arrow[r] & \overset{0}{\tilde{H}_{i+1}(U_+) \oplus \tilde{H}_{i+1}(U_-)} \arrow[r] & \overset{\tilde{H}_{i+1}(S^n)}{\tilde{H}_{i+1}(U_+ \cup U_-)} \arrow[dll, rounded corners, to path={[pos=0.25] -- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-4ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & \underset{\tilde{H}_i(S^{n-1})}{\tilde{H}_i(U_+ \cap U_-)} \arrow[r] & \underset{0}{\tilde{H}_i(U_+) \oplus \tilde{H}_i(U_-)} \arrow[r] & \tilde{H}_i(U_+ \cup U_-)
\end{tikzcd}$$

So $\tilde{H}_{i+1}(S^n) = \tilde{H}_i(S^{n-1})$

$\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & \text{if } *=n \\ 0 & \text{o.w.} \end{cases}$

def. notation we can generate $H_n(S^n)$ by $[S^n]$

$p: U_+ \cap U_- \to S^{n-1} \quad (x_1, \cdots x_{n+1}) \mapsto (x_2, \ldots x_{n+1}) \Rightarrow p_* \partial [S_n] = [S_{n-1}]$

week3 lecture 1

lemma : turn a SES into a LES

Suppose that we have a commuting diagram of chain cxs, chain maps, rows are SES.

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f_A"] \arrow[ur] & B \arrow[r, "\pi"] \arrow[d, "f_B"] \arrow[ur] & C \arrow[r] \arrow[d, "f_C"] \arrow[ur] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] \arrow[ur] & B' \arrow[r, "\pi'"'] \arrow[ur] & C' \arrow[r] \arrow[ur] & 0
\end{tikzcd}$$

then you get a commuting diagram of LES

$$\begin{tikzcd}
{} \arrow[r] & H_i(B) \arrow[r] \arrow[d, "f_{B*}"] \arrow[ur] & H_i(C) \arrow[r, "\partial"] \arrow[d, "f_{C*}"] \arrow[ur] \arrow[dr, bend left] \arrow[dr, bend right] & H_{i-1}(A) \arrow[r] \arrow[d, "f_{A*}"] \arrow[ur] & {} \\
{} \arrow[r] & H_i(B') \arrow[r] \arrow[ur] & H_i(C') \arrow[r, "\partial'"'] \arrow[ur] & H_{i-1}(A') \arrow[r] \arrow[ur] & {}
\end{tikzcd}$$

Proof: Prove the commutativity of red square.

let $[c] \in H_i(C)$ so $dc=0$.

↳ first find a, b. π surjective, $b \in B$, have $\pi(b)=c$. $\pi d b = d\pi b = d(c) = 0$ so $db \in \ker \pi = \mathrm{Im}(i)$

so $\exists a \in A$, $i(a)=db$. Set $\partial[c]=[a]$.

↳ let $a' = f_A(a)$, $b' = f_B(b)$, $c' = f_C(c)$

↳ show $\pi'(b') = c'$

indeed $\pi'(b') = \pi'(f_B(b)) = f_C(\pi(b)) = f_C(c) = c'$

↳ show $i'(a') = db'$

Indeed, $i'(a') = i'(f_A(a)) = f_B(i(a)) = f_B(db) = d f_B(b) = db'$

↳ so to find $\delta([c'])$, have b' for $c' = \pi'(b')$ and $i'(a') = db'$ so that $\delta([c']) = [a']$.

↳ then, $\partial' f_{C*}[c] = \partial'[c'] = [a'] = [f_A(a)] = f_{A*}\partial[c]$.

so the square commutes

☐

Proof scheme

↳
$$\begin{tikzcd}
\arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & {} \\
\arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & {}
\end{tikzcd}$$
get
$$\begin{tikzcd}
H(B) \arrow[r] \arrow[d] & H(C) \arrow[r, "\partial"] \arrow[d] & H(A) \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

↳ set $\partial[c] = [a]$

↳ show $\pi'(b) = c'$ and $i'(a) = db'$

↳ $\partial'[c'] = [a']$ ⇒ commute.

Idea: look at the other unproven boxes.

Example two short MVS into two long MVS

let $f: X \to Y$ with $Y = U_1 \cup U_2$, U_1, U_2 open.

then, $V_1 = f^{-1}(U_1)$, $V_2 = f^{-1}(U_2)$ and $X = V_1 \cup V_2$. Then $f_\#$ induces

$$\begin{tikzcd}
0 \arrow[r] & C_*(V_1 \cap V_2) \arrow[r] \arrow[d, "f_\#"] & C_*(V_1) \oplus C_*(V_2) \arrow[r] \arrow[d, "f_\#"] & \overset{v}{C_*}(V_1 \cup V_2) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C_*(U_1 \cap U_2) \arrow[r] & C_*(U_1) \oplus C_*(U_2) \arrow[r] & \overset{u}{C_*}(U_1 \cup U_2) \arrow[r] & 0
\end{tikzcd}$$

then, we can get the LES of this above using the above prop.

Prop: $\underline{r_{n*}: \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)}$ is defined by $[S^n] \to -[S^n]$.

$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[r] \arrow[d, phantom, "\wr"] & \tilde{H}_n(S^n) \arrow[d, phantom, "\wr"] \\ \langle [S^n] \rangle & \langle [S^n] \rangle \end{tikzcd}$$

$r_n = S^n \to S^n$

$(x_1, x_2, \cdots, x_{n+1}) \mapsto (x_1, \cdots x_n, -x_{n+1})$ $\quad S^n = U_+ \cup U_-$, $r: U_+ \to U_+$, $U_- \to U_-$

Proof : induction on n:

???

$n=0$, $\quad [S^0] = \boxed{[\sigma_1 - \sigma_{-1}]}$ $\quad \sigma_1 - \sigma_{-1} \in \ker(d)$

$r_{0*}[S^0] = r_{0*}[\sigma_1 - \sigma_{-1}] = [-\sigma_1 + \sigma_{-1}] = -[S^0]$.

$n > 0$,

now, consider the map that r_n induces on (S^n, U_+, U_-)

the SES is:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{C}_*(U_+ \cap U_-) \arrow[r] \arrow[d, "r_{n\#}"] & \tilde{C}_*(U_+) \oplus \tilde{C}_*(U_-) \arrow[r] \arrow[d, "r_{n\#}"] & \tilde{C}_*(U_+ \cup U_-) \arrow[r] \arrow[d, "r_{n\#}"] & 0 \\ 0 \arrow[r] & \tilde{C}_*(U_+ \cap U_-) \arrow[r] & \tilde{C}_*(U_+) \oplus \tilde{C}_*(U_-) \arrow[r] & \tilde{C}_*(U_+ \cup U_-) \arrow[r] & 0 \end{tikzcd}$$

get the LES by the above purposition

$$\begin{tikzcd} \arrow[r] & \cancel{\tilde{H}_i(U_+) \oplus \tilde{H}_i(U_-)}^{\,0} \arrow[r] \arrow[d, "r_{n*}"] & \tilde{H}_i(U_+ \cup U_-) \arrow[r, "\partial", "\cong"'] \arrow[d, "r_{n*}"] & \tilde{H}_{i-1}(U_+ \cap U_-)^{\,S^{n-1}} \arrow[r] \arrow[d, "r_{n*}"] & \cancel{\tilde{H}_{i-1}(U_+) \oplus \tilde{H}_{i-1}(U_-)}^{\,0} \arrow[r] \arrow[d, "r_{n*}"] & {} \\ \arrow[r] & \cancel{\tilde{H}_i(U_+) \oplus \tilde{H}_i(U_-)}^{\,0} \arrow[r] & \tilde{H}_i(U_+ \cup U_-) \arrow[r, "\partial", "\cong"'] & \tilde{H}_{i-1}(U_+ \cap U_-)^{\,S^{n-1}} \arrow[r] & \cancel{\tilde{H}_{i-1}(U_+) \oplus \tilde{H}_{i-1}(U_-)}^{\,0} \arrow[r] & {} \end{tikzcd}$$

consider $p: U_+ \cap U_- \to S^{n-1}$

$(x_1, \cdots x_{n+1}) \mapsto (x_2, \cdots, x_{n+1})$

has $p \circ r_n = r_{n-1} \circ p$. (flip last, cut 1st vs cut 1st, flip 2nd last)

so extend it to

$$\begin{tikzcd} \tilde{H}_i(U_+ \cup U_-) \arrow[r, "\partial", "\cong"'] \arrow[d, "r_{n*}"] & \tilde{H}_{i-1}(U_+ \cap U_-)^{\,S^{n-1}} \arrow[r, "p"] \arrow[d, "r_{n*}"] & \tilde{H}_{i-1}(S^{n-1}) \arrow[d, "r_{(n-1)*}"] \\ \tilde{H}_i(U_+ \cup U_-) \arrow[r, "\partial", "\cong"'] & \tilde{H}_{i-1}(U_+ \cap U_-)^{\,S^{n-1}} \arrow[r, "p"] & \tilde{H}_{i-1}(S^{n-1}) \end{tikzcd}$$

$r_{(n-1)*}[S^{n-1}] = -[S^{n-1}]$

$r_{n*}[S^n] = -[S^n]$

cor. let $r \in S^n$, let $r_v : S^n \to S^n$ be reflectin across the plane $\perp$ to v.

$\Rightarrow$ $r_{v*}([S^n]) = -[S^n]$

Proof: S^n is p.c. If γ is path from v to $\overset{(0,0,\dots,0,1)}{e_{n+1}}$, $V_{\gamma(v)}$ is a homotopy from r_v to $r_{e_{n+1}} = r_n$. So $r_{v*} = r_{n*}$.

Excision + Collapsing of a pair

def. Deformation retract

let $A \subset Z$ then A is a d.r. of Z if $\exists$ $p : (Z,A) \to (Z,A)$

s.t. $p \circ i : (A,A) \to (A,A) = 1_{(A,A)}$ $\qquad$ $i : (A,A) \hookrightarrow (X,A)$ is inclusin

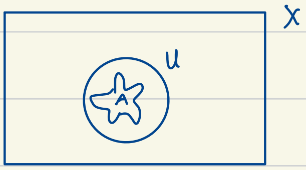
$i \circ p : (Z,A) \to (Z,A) \underset{\uparrow}{\sim} 1_{(Z,A)}$

homotopy map of pairs

$\Rightarrow Z \sim A$.

def good pair

let (X,A) be pair of spaces. It's a good pair if $\exists\, U \subset X$ open s.t. (U,A) is a d.r.



e.g. Submanifold is a good pair but $\mathbb{Z}, \mathbb{Q}$ is not.

Thm The good pair isomorphism

Suppose (X,A) is a good pair, $\pi : (X,A) \to (X/A, A/A)$ then

$\pi_* : H_*(X,A) \to H_*(X/A, A/A) \underset{\uparrow}{\cong} \tilde{H}_*(X/A)$ is an isomorphism

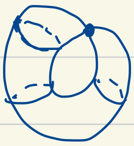
b/c $H_*(X, \{p\}) = \tilde{H}_*(X)$

??? Proof ?

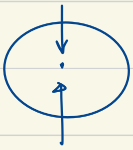
Example 1 of computing using good pair hom.

$X=S^2$, $A=\{n,s\}$. $Z=X/A$

Thm states $H(X,A)=H(X/A,A/A)=\tilde{H}(X/A)$



or



so, using LES, get

note LES of pair works on all ~ (why) ← NTS

$$\begin{tikzcd}
\overset{=0}{\tilde{H}_2(A)} \arrow[r] & \overset{=\mathbb{Z}\ \cong}{\tilde{H}_2(X)} \arrow[r] & \overset{\text{so}\ \boxed{=\mathbb{Z}}}{\tilde{H}_2(X,A)} \arrow[dll] \\
\overset{=0}{\tilde{H}_1(A)} \arrow[r] & \overset{=0}{\tilde{H}_1(X)} \arrow[r] & \overset{\boxed{\cong\mathbb{Z}}}{\tilde{H}_1(X,A)} \arrow[dll, "\cong"] \\
\overset{=\mathbb{Z}}{\tilde{H}_0(A)} \arrow[r, "\text{map is } 0"'] & \overset{=0}{\tilde{H}_0(X)} \arrow[r] & \overset{\boxed{=0}}{\tilde{H}_0(X,A)} \arrow[r] & 0
\end{tikzcd}$$

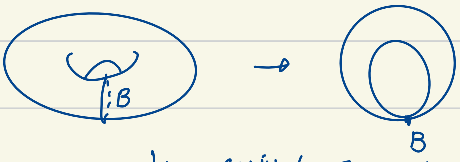
b/c path connected.

$$H_*(A)=\begin{cases}\tilde{H}_*(A) & *\neq 0\\ \tilde{H}_0(A)\oplus\mathbb{Z} & *=0\end{cases}$$

note: $H_*(X,A)=\tilde{H}_*(X,A)$

so $$H_*(Z)=H_*(X/A)\cong H_*(X,A)\cong\tilde{H}_*(X,A)=\begin{cases}\mathbb{Z} & \text{if } *=1,2\\ 0 & \text{o.w.}\end{cases}$$

Ex2. $Y=S^1\times S^1$, $B=S^1\times 1$



hom equiv to Z in ex1.

we know $H_*(B)$, $H_*(Y,B)\cong H_*(Z)$, WTS $H_*(Y)$.

two steps

1) show $i_{1*}: S^1\to S^1\times S^1$ is injective (which helps with computation for LES)

2) show that $Y/B\cong Z$.

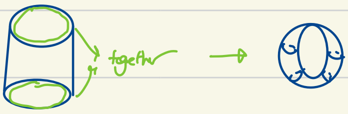
1) let
$$\begin{cases} i_1: S^1\to S^1\times S^1\\ \quad x\mapsto(x,1)\\ i_2: S^1\to S^1\times S^1\\ \quad x\mapsto(1,x)\\ \pi_1: S^1\times S^1\to S^1\\ \quad (x,y)\to x\\ \pi_2: S^1\times S^1\to S^1\\ \quad (x,y)\to y\end{cases}$$

$\pi_1\circ i_1=1_{S^1}$ $\Rightarrow$ $\pi_{1*}\circ i_*=1_{H_*(S^1)}$ $\Rightarrow$ i_* is injective

$\pi_2\circ i_2=1_{S^1}$ $\Rightarrow$ $\pi_{2*}\circ i_{2*}=1_{H_*(S^1)}$

2) $Y/B\cong Z$

$Z=S^1\times[-1,1]/S^1\times S^0$



$S^1\times[-1,1]\to S^2\to Z$

$S^1\times[-1,1]\to T^2\to Z$

using LES of pair $H_*(T^2)$. have $0 \to C_*(B) \to C_*(T^2) \to \overset{\overset{Z}{IS}}{C_*(T^2/B)} \to 0$

$$\begin{tikzcd}
\overset{=0}{\tilde{H}_2(B)} \arrow[r] & \tilde{H}_2(T^2) \arrow[r] & \tilde{H}_2(T^2/B) \overset{}{=\mathbb{Z}} \arrow[dll, "\partial=0"'] \\
\tilde{H}_1(B) =\mathbb{Z} \arrow[r, "\text{injective as shown earlier.}"] & \tilde{H}_1(T^2) \arrow[r] & \tilde{H}_1(T^2/B) =\mathbb{Z} \arrow[dll] \\
\tilde{H}_0(B) =0 \arrow[r] & \tilde{H}_0(T^2) =\mathbb{Z} \arrow[r] & \tilde{H}_0(T^2/B) =0
\end{tikzcd}$$

So break up into two

$$0 \to \underset{\uparrow =\mathbb{Z}}{\tilde{H}_2(T^2)} \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \to \underset{=\mathbb{Z}\oplus\mathbb{Z}}{\tilde{H}_1(T^2)} \to \mathbb{Z} \to 0$$

so $\tilde{H}_i(T^2) = \begin{cases} \mathbb{Z} & i=2 \\ \mathbb{Z}^2 & i=1 \\ 0 & \text{o.w.} \end{cases}$ or $\tilde{H}_i(T^2) = \begin{cases} \mathbb{Z} & i=0,2 \\ \mathbb{Z}^2 & i=1 \\ 0 & \text{o.w.} \end{cases}$

week 3 lecture 2

thm: the five lemma (helps to prove excision).

Given commutative diagram of R modules.

1) each row are exact

2) $f_{i\pm1}, f_{i\pm2}$ are iso

$$\begin{tikzcd}
{} \arrow[r] & A_{i-2} \arrow[r] \arrow[d, "f_{i-2}"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & A_i \arrow[r] \arrow[d, "f_i"] & A_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & A_{i+2} \arrow[r] \arrow[d, "f_{i+2}"] & {} \\
{} \arrow[r] & B_{i-2} \arrow[r] & B_{i-1} \arrow[r] & B_i \arrow[r] & B_{i+1} \arrow[r] & B_{i+2} \arrow[r] & {}
\end{tikzcd}$$

def $C_*^{\mathcal{U}}(X,A)$

let $\mathcal{U} = \{U_j \mid j \in J\}$ be an open cover for X.

then if $A \subset X$, $\mathcal{U}_A = \{U_j \cap A \mid j \in J\}$ is an open cover of A.

and $C_*^{\mathcal{U}_A}(X) \subset C_*^{\mathcal{U}}(X)$

define $C_*^{\mathcal{U}}(X,A) = C_*^{\mathcal{U}}(X) \Big/ C_*^{\mathcal{U}_A}(X)$

the map $i: C_*^{\mathcal{U}}(X) \to C_*(X)$

induces $i: C_*^{\mathcal{U}}(X,A) \to C_*(X,A)$

lemma: $i_*: H_*^{u}(X,A) \to H_*(X,A)$ is an isomorphism

induced by $i: C_*^{u}(X,A) \to C_*(X,A)$

We have

$$\begin{tikzcd}
0 \arrow[r] & C_*^{u}(A) \arrow[r] \arrow[d, "i"] & C_*^{u}(X) \arrow[r] \arrow[d, "i"] & C_*^{u}(X,A) \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X,A) \arrow[r] & 0
\end{tikzcd}$$

so we get commuty diagram of LES

$$\begin{tikzcd}
{} \arrow[r] & H_*^{u}(A) \arrow[r] \arrow[d, "i_*"] & H_*^{u}(X) \arrow[r] \arrow[d, "i_*"] & H_*^{u}(X,A) \arrow[r] \arrow[d, "i_*"] & H_{*-1}^{u}(A) \arrow[r] \arrow[d, "i_*"] & H_*^{u}(C) \arrow[r] \arrow[d, "i_*"] & {} \\
{} \arrow[r] & H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X,A) \arrow[r] & H_{*-1}(A) \arrow[r] & H_*(C) \arrow[r] & {}
\end{tikzcd}$$

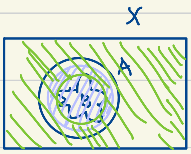
pink are iso ⇒ blue is iso.

Thm (Excision)

Suppose $B \subset A \subset X$ and $\bar{B} \subset \text{int}(A)$. Letting $j: (X-B, A-B) \to (X,A)$ have

$j_*: H_*(X-B, A-B) \to H_*(X,A)$ is an $\cong$

Proof.: note that $\mathcal{U} = \{\text{int}(A), X-\bar{B}\}$ is an op. cov. for X.



notation: if $\sigma: \Delta^k \to X$, write $\sigma \triangleleft \mathcal{U}$ if $\text{im}\,\sigma \subset \mathcal{U}_j$ for some j.

then $C_*^{u}(X) = \langle \sigma \mid \sigma \triangleleft \mathcal{U} \rangle$

$= \langle \sigma \mid \text{im}(\sigma) \cap B = \emptyset \rangle \oplus \langle \sigma \mid \text{im}(\sigma) \cap B \neq \emptyset \rangle$

$= C_*^{u}(X-B) \oplus M_B$ $\qquad M_B = \boxed{\langle \sigma \mid \text{im}(\sigma) \subset A \text{ and } \text{im}(\sigma) \cap B \neq \emptyset \rangle.}$

Similarly, $C_*^{u_A}(A) = C_*^{u}(A-B) \oplus M_B$

either $\text{im}(\sigma) \subset X-B$ or

$\text{im}(\sigma) \not\subset X-B$

$\exists x \in \text{im}(\sigma)$ s.t. $x \notin X-B$ so $\text{im}(\sigma) \cap B \neq \emptyset$

$\Rightarrow x \in B$

$u \subseteq \text{int}(A) \Rightarrow \text{im}(\sigma) \subset A$

$u \subseteq X-\bar{B} \subseteq X-B$ so $\text{im}(\sigma) \cap B = \emptyset$.

note: If $C' \subset C$, then the inclusion is an iso

$$\frac{C}{C'} \to \frac{C \oplus M}{C' \oplus M} \text{ is an iso}$$

so that $C = C_*^{u}(X-B)$ then $j_\#: \dfrac{C_*^{u}(X-B)}{C_*^{u_A}(A-B)} \to \dfrac{C_*^{u}(X-B) \oplus M_B}{C_*^{u_A}(A-B) \oplus M_B} = \dfrac{C_*^{u}(X)}{C_*^{u_A}(A)}$ is an iso

$C' = C_*^{u_A}(A-B)$

so $j_\#: C_*^{u}(X-B, A-B) \to C_*^{u}(X,A)$ is an iso

$j_*: H_*^{u}(X-B, A-B) \to H_*^{u}(X,A)$ is an iso (because H depends on C) now,

$$\begin{tikzcd} H_*^u(X-B, A-B) \arrow[r, "j_* \cong"] \arrow[d, "i_*"] & H_*^u(X,A) \arrow[d, "i_*"] \\ H_*(X-B, A-B) \arrow[r, "j^*"] & H_*(X,A) \end{tikzcd}$$

By our prev lemma, which is equal to five lemma + subdiv lemma.

hence, $j_*: H_*(X-B, A-B) \to H_*(X,A)$ is an iso,

Proof scheme:

↳ WTS: $H_*(X-B, A-B) \cong H_*(X,A)$ where $B \subset A \subset X$, $\bar{B} \subset int(A)$

↳ note that $\mathcal{U} = \{Int(A), X-\bar{B})$ is an open cover

↳ write $C_*^u(X) = C_*^u(X-B) \oplus M_B$
$C_*^u(A) = C_*^u(A-B) \oplus M_B$ } $C \subset C' \Rightarrow \frac{C'}{C} \cong \frac{C' \oplus M_B}{C \oplus M_B}$

↳ $\frac{C_*^u(X-B)}{C_*^u(A-B)} \cong \frac{C_*^u(X)}{C_*^u(A)}$

↳ $C_*^u(X-B, A-B) \cong C_*^u(X,A) \Rightarrow$ use lemma, get $H_*(\quad) \cong H_*(\quad)$

Prop (LES of a triple)

Suppose $X \subset Y \subset Z$ then there's a LES: to remember, each $(a,b) \subset (c,d)$ $a \subset c$, $b \subset d$.

$$H_*(Y,X) \xrightarrow{j_{1*}} H_*(Z,X) \xrightarrow{j_{2*}} H_*(Z,Y) \xrightarrow{\partial} H_{*-1}(\quad) \to \cdots$$

proof group theory says below is a SES

$$0 \to \frac{C_*(Y)}{C_*(X)} \to \frac{C_*(Z)}{C_*(X)} \to \frac{C_*(Z)}{C_*(Y)} \to 0 \qquad C \cong B/A$$

$$0 \to C_*(Y,X) \to C_*(Z,X) \to C_*(Z,Y) \to 0$$

then use snake lemma to get the result.

lemma: deformation retraction induces iso on homology.

let A be a d.r. of $\mathcal{U}$. let $i: (X,A) \to (X,\mathcal{U})$ be the inclusion map.

then $i_*\ H_*(X,A) \to H_*(X,\mathcal{U})$ is an iso.

Proof. using LES of $(\mathcal{U},A)$ then LES of triple $(X,\mathcal{U},A)$.

$$\cdots \longrightarrow H_*(A) \underset{\cong}{\xrightarrow{i_*}} H_*(\mathcal{U}) \xrightarrow{f_*} H_*(\mathcal{U},A) \xrightarrow{g_*} H_{*-1}(A) \underset{\cong}{\xrightarrow{i_*}} H_{*-1}(\mathcal{U}) \longrightarrow \cdots$$

$ker(f_*) = Im(H_*) = H_*(\mathcal{U})$ so f_* is the 0 map.

$Im(g_*) = ker(i_*) = 0$ so g_* is the 0 map.

$\Rightarrow f_*, g_*$ are the 0 map, so $H_*(\mathcal{U},A) \cong 0$

(X, U, A) $A \subset U \subset X$

$$\begin{tikzcd} \cdots \arrow[r] & \underset{=0}{H_*(U,A)} \arrow[r, "j_{1*}"] & H_*(X,A) \arrow[r, "j_{2*}", "\uparrow \cong"'] & H_*(X,U) \arrow[r, "\partial"] & \underset{=0}{H_{*-1}(U,A)} \arrow[r, "j_{1*}"] & H_{*-1}(X,A) \arrow[r] & \cdots \end{tikzcd}$$

so $H_*(X,A) \cong H_*(X,U)$

def good pair:

(X,A) is good pair if $\begin{cases} \exists\, U \subset X \text{ open} \\ \bar{A} \subset U \\ A \text{ is a d.r. of } U. \end{cases}$

excision is inclusion $H_*(X-B, A-B) \xrightarrow{i_*} H_*(X,A)$

collapsing is quotient $H_*(X,A) \xrightarrow{\pi_*} H_*(X/A, A/A)$

thm collapsing of a pair.

Suppose (X,A) is a good pair, and let $\pi_*: H_*(X,A) \to \overset{\substack{\tilde{H}_*(X/A) \\ \cong}}{H_*(X/A, A/A)}$ is an iso.

Proof idea: extend the commutative diagram downwards

$$H_*(X-A, U-A) \xrightarrow{j_*} H_*(X,U) \xleftarrow{i_*} H_*(X,A)$$

j's, i's, π's,

Proof:

$$\begin{tikzcd} H_*(X-A, U-A) \arrow[r, "j_*", "\text{excision} \Rightarrow \cong"'] \arrow[d, "\pi_{1*}"] & H_*(X,U) \arrow[d, "\pi_{2*}"] & H_*(X,A) \arrow[l, "i_*"', "\text{d.r.} \Rightarrow \cong"] \arrow[d, "\boxed{\pi_{3*}}"] \\ H_*(X/A - A/A, U/A - A/A) \arrow[r, "j_*"', "\text{excision} \Rightarrow \cong"] & H_*(X/A, U/A) & H_*(X/A, A/A) \arrow[l, "i_*", "\text{d.r.} \Rightarrow \cong"'] \end{tikzcd}$$

π_1 is a homeo ⇒ π_2 is ⇒ π_3 is.

$\pi_1: (X-A, U-A) \to (X/A - A/A, U/A - A/A)$ is a homeo.

you can recov LHS from RHS.

Def. manifold

A space X is a manifold if it's

1) metrisable (Hausdorff, second countable)

2) every $x \in X$ has an open nbhd $U_x \cong \mathbb{R}^n$

$H(X)$ is only one that has only 1 copy of $\mathbb{Z}$ at 0.

$H(X,A)$ or $\tilde{H}$ all don't have this problem

thm H_* of manifold

If X is a mani, and $x \in X$ then $H_*(X, X-x) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{o.w.} \end{cases}$

Proof scheme:
- use excision: $\Rightarrow$ $H_*(X, X-\{p\}) \cong H_*(D^n, D^n - \{p\})$
- use LES of pair to find out what RHS really is.

Proof. pick $U_x \subset X$ as above, then $U_x \cong \mathbb{R}^n$, $x \mapsto 0$ $\quad D^n \subset \mathbb{R}^n \cong U_x \subset X$

So $D^n \subset X$.

By excision, $(B \subset A \subset X,\ \bar{B} \subset \text{int}(A)) \Rightarrow H_*(X-B, A-B) \to H_*(X, A)$ is $\cong$

$A = X - \{p\}$
$B = X - D^n$

$\Rightarrow H_*(D^n, D^n - p) \to H_*(X, X-\{p\})$ is $\cong$

i.e. excision with $\begin{cases} X = X \\ A = X-\{p\} \\ B = X - D^n \end{cases}$

get $H_*(X, X-p) \cong H_*(D^n, D^n - \{p\})$
$\cong H_*(D^n, S^{n-1})$

use LES of pair on $H_*(D^n, S^{n-1})$, and using $\tilde{H}_*$, and note $\tilde{H}_*(D^n) = 0$

$$\underset{=0}{\tilde{H}_*(D^n)} \xrightarrow{\pi_*} \tilde{H}_*(D^n, S^{n-1}) \xrightarrow[\cong]{\partial} \tilde{H}_{*-1}(S^{n-1}) \xrightarrow{i} \underset{=0}{\tilde{H}_{*-1}(D^n)}$$

So $\tilde{H}_*(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{o.w.} \end{cases}$
$\qquad \| $
$H_*(D^n, S^{n-1})$

Cor if M^m, N^n are m and n manifolds and $M \cong N$ then $m = n$.

week 3 lec 3

(II) Cellular homology.

def ~~degree of S^n maps~~

recall $H_n(S^n) \cong \mathbb{Z}$ generated by $[S^n]$.

then, if $f: S^n \to S^n$ has $f_*: [S^n] \mapsto k[S^n]$ $k \in \mathbb{Z}$, then its degree is k.

Properties

1) $\deg(1_{S^n}) = 1$, 2) $\deg(f \circ g) = (\deg f)(\deg g)$ 3) $f_0 \sim f_1 \Rightarrow \deg(f_1) = \deg(f_0)$
4) $f: S^n \xrightarrow{\sim} S^n$ then $\deg(f) = \begin{cases} 1 & \text{orientation preserving} \\ \text{or} & \\ -1 & \text{orientation reversing} \end{cases}$

5) if $r_*: S^n \to S^n$ is reflection $v^\perp$, $\deg r_v = -1$

6) if $A: S^n \to S^n$ antipodal map, then $\deg(A) = (-1)^{n+1}$

Cor: Antipodal map $\not\simeq 1_{S^n}$ if n even.

Local degree

If $p \in S^n$, $S^n - p \cong D^n$ which is contractible

then, $\pi_* : H_n(S^n) \to H_n(S^n, S^n - p)$ is $\cong$ for $n \geq 1$.

defn : $[S^n, S^n - p]$ & $[U, U-p]$

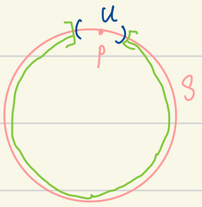
define $[S^n, S^n-p]$ by $\pi_*[S^n] = [S^n, S^n-p]$: element in $H_n(S^n, S^n-p)$ induced by $\pi_*[S^n]$.

Similarly if $U \subset S^n$ is open, if $p \in U$, let $B = S^n \setminus U$, B is closed.

$\bar{B} \subset \text{int}(S^n - p) \Rightarrow (S^n - B, S^n - B - p) = (U, U-p)$ (recall excision: $\bar{B} \subset \text{int}(A) \Rightarrow (X-B, A-B) \to (X, A)$ is iso

So we can use excision $j_* : H_n(U, U-p) \to H_n(S^n, S^n - p)^{=\mathbb{Z}}$ is an $\cong$.

define $[U, U-p]$ by $[U, U-p] \to [S^n, S^n-p]$



defn. $[U', U'-p] \to [U, U-p]$ is an $\cong$.

If $p \in U' \subset U$, we have a commutative diagram

$$\begin{tikzcd}
H_n(U, U-p) \arrow[rr, "{[U,U-p] \mapsto [S^n,S^n-p]}", "\cong"'] & & H_n(S^n, S^n-p) \\
H_n(U', U'-p) \arrow[u, "{i_* : [U',U'-p] \to [U,U-p] \text{ is an } \cong}"] \arrow[urr, "\cong", "{[U',U'-p] \mapsto [S^n, S^n-p]}"'] & &
\end{tikzcd}$$

Because the two other ones are $\cong$ by excision, i_* is an iso.

defn Local degree of a map

If $f : S^n \to S^n$, let $p \in S^n$, and $f^{-1}(p) = \{q_1, \cdots q_n\}$ is finite, then

S^n Hausdorff: fnd $U_i \subset S^n$ open, s.t. $U_i \cap U_j = \phi$, if $i \neq j$ and $q_i \in U_i$ and $(q_i \in U_j \Leftrightarrow i = j)$

$f_i : (U_i, U_i - q_i) \to (S^n, S^n - p)$ (not necessarily inclusion, it can do weird stuff)

so $f_{i*}[U_i, U_i - q_i] = k_i[S^n, S^n - p]$

So k_i is the local degree of f at q_i. denoted $\deg_{q_i} f = k_i$

remember: $i_n : [U_i, U_i - q] \hookrightarrow [S, S-q]$
$f \circ i : [U_i, U_i - q] \hookrightarrow \deg_{q_i}[S, S-q]$

lemma: $\deg_{q_i} f = k_i$ does not depend on the choice of U_i

Proof: Suppose that $U_i' \subset U_i$, $q_i \in U_i'$ then

$$\begin{tikzcd}
H_n(U_i, U_i - q_i) \arrow[r, "f_*"] & H_n(S^n, S^n - q) \\
H_n(U_i', U_i' - q_i) \arrow[u, "i_*"] \arrow[ur, "f_*'"'] &
\end{tikzcd}$$

$i_* [U_i', U_i' - p] = [U_i, U_i - p]$ so $\deg f_* = \deg f_*'$.

generally, given U_i, U_i' two opn nbhd of p_i, so is $U_i \cap U_i'$ so $U_i \cap U_i'$, U_i, U_i' gives the same degrees.

Rmk. homology on the union of open sets.

let $V = \coprod_i U_i \subset S^n$. it's open. By excision, $(\underset{\underset{V}{\|}}{S^n - (S^n - V)}, \underset{\underset{V - f^{-1}(p)}{\|}}{S^n - f^{-1}(p) - (S^n - V)})$ & $(S^n, S^n - f^{-1}(p))$

get $j_* : H_n(V, V - f^{-1}(p)) \xrightarrow{\cong} H_n(S^n, S^n - f^{-1}(p))$

$$\begin{array}{c} H_n(V, V - f^{-1}(p)) \\ \cong \\ \bigoplus H_n(U_i, U_i - p_i) \cong \mathbb{Z}^r \end{array}$$

so that $[U_i, U_i - p_i]$ form a basis for $H_n(S^n, S^n - f^{-1}(p))$.

lem. map $H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p))$

let map $H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p)) \cong \bigoplus H_n(U_i, U_i - q_i)$

given by $[S^n] \longmapsto \sum_{i=1}^{r} [U_i, U_i - q_i]$

Proof. there's a commutative diagram

note that $H_n(\underbrace{V, V - q_j}_{\substack{\text{all components are contractible} \\ \text{except for the } j\text{th}}}) \cong H_n(U_j, U_j - q_j)$

$$\begin{tikzcd}
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] & H_n(S^n, S^n - q_j) \\
H_n(V, V - f^{-1}(p)) \arrow[r] \arrow[u, "\cong"', "j_* \text{ (excision above)}"] & H_n(V, V - q_j) \cong H_n(U_j, U_j - q_j) \arrow[u, "\cong"', "j_* \text{ (by excision also)}"]
\end{tikzcd}$$

Vertical maps are isomorphisms. So we reverse the two vertical arrow and add things to diagram.

$$\begin{tikzcd}
H_n(S^n) \arrow[d] \arrow[dr] \arrow[ddrr, bend left, "\alpha"] \arrow[ddd, bend right=60, "\beta"'] & & \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] \arrow[d, "j_*^{-1}"', "\cong"] & H_n(S^n, S^n - q_j) \arrow[d, "j_*^{-1}"', "\cong"] & \\
H_n(V, V - f^{-1}(p)) \arrow[r, "\pi"] \arrow[d, equal] & H_n(V, V - q_j) \arrow[r, phantom, "\cong"] & H_n(U_j, U_j - q_j) \\
\oplus H_n(U_i, U_i - q_i) \arrow[urr, "\pi_j"'] & &
\end{tikzcd}$$

diagram commutes so $\alpha = \beta \circ \pi_j$

$\alpha([S^n]) = [U_j, U_j - q_j]$

$\Rightarrow \beta \circ \pi_j([S^n]) = [U_j, U_j - q_j]$

so $\beta([S^n]) = \sum_{j=1}^{n} [U_j, U_j - q_j]$

Proof scheme

- Start with square
$$\begin{tikzcd}
(S^n, S^n - f^{-1}(p)) \arrow[r] \arrow[d] & (S^n, S^n - q_i) \arrow[d] \\
(V, V - f^{-1}(p)) \arrow[r] & (V, V - q_i)
\end{tikzcd}$$
- reverse vertical arrows, add $[S^n]$ on top and two isos.
- Commutativity shows desired result.

Thm. degree of f as sum of local degrees

Spose $f: S^n \to S^n$, $f^{-1}(p) = \{q_1, q_2, \cdots q_r\}$.

then, $\deg f = \sum_{i=1}^{r} \deg_{q_i} f$.

Proof

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] \arrow[dr, bend left=70, "\alpha"] \arrow[dd, bend right, "\beta \text{ is } j_*^{-1}"'] \arrow[dr, bend right=90, "\gamma"'] & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r, "f_*"] & H_n(S^n, S^n - p) \\
\oplus H_n(U_i, U_i - q_i) \arrow[ur, "\oplus f"'] &
\end{tikzcd}$$

$\alpha: [S^n] \to \deg f\,[S^n] \to \deg f\,[S^n, S^n - p]$

$\gamma: [S^n] \xrightarrow{\beta} \sum_{i=1}^{r} [U_i, U_i - q_i]$

$\to \sum_{i=1}^{r} (\deg_{q_i} f)\,[S^n, S^n - p]$

so $\deg f = \sum_{i=1}^{r} (\deg_{q_i} f)$

??? example about homeomorphism <u>not understand</u> !!!

Motivation section is missed!

To remember for local degree stuff.

1. define local degree :

$i: [U_i, U_i - q_i] \hookrightarrow [S^n, S^n - q_i]$ excision

degree is degree of $f \circ i$

2. lemma: show $H_n([S^n]) \cong \bigoplus_i H_n([U_i, U_i - q_i])$

this point, nothing to do with H.

expand on

$$\begin{tikzcd}
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] \arrow[d] & H^n(S^n, S^n - q_i) \arrow[d] \\
H_n(V, V - f^{-1}(p)) \arrow[r, "\pi"] & H^n(V, V - q_i)
\end{tikzcd}$$

3. include the f now, show that

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & {} \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[d, phantom, "\cong" rotate=90] & \\
{} &
\end{tikzcd}$$

week 4 lec 1

The cellular chain complex.

def. attaching along a function

let $B \subset Y$, assume $f: B \to X$. Then $X \cup_f Y = X \amalg Y / \sim$ where $\sim$ is the smallest equivalence relation s.t. $b \sim f(b)$ $\forall b \in B$. So this space is obtained by gluing X to Y along f.

def Attaching a k-cell

$(Y, B) = (D^k, S^{k-1})$ then $X \cup_f D^k$ is attaching a k-cell to X.

def. finite cell complex

A finite cell complex (fcc) is a space X equipped with closed subsets

$$\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \cdots \subset X_n$$

where X_k is obtained from X_{k-1} by attaching finite k-cells. s.t.

(X_k is called k-skeleton.) where

$$X_k \cong X_{k-1} \cup_F \coprod_{\alpha \in A_k} D^k$$

$$F: \coprod_{\alpha \in A_k} S^{k-1} \to X_{k-1}, \quad F = \coprod_{\alpha \in A} F_\alpha, \quad F_\alpha : S^{k-1} \to X_{k-1}$$

def open sets of infinite conditions

$X = \bigcup_{i=1}^{\infty} X_i$ then $U \subset X$ open $\Leftrightarrow$ $U \cap X_k$ is open for all k.

Examples of fcc complex constructions

- graph : $\begin{cases} V \text{ of } & 0\text{-cells,} \\ e \text{ of } & 1\text{-cells.} \end{cases}$
- S^k : $\begin{cases} 1 \text{ of } & 0\text{-cell} \\ 1 \text{ of } & k\text{-cell} \end{cases}$
- D^{k+1} : $\begin{cases} 1 \text{ of } & 0\text{-cell} \\ 1 \text{ of } & k\text{-cell} \\ 1 \text{ of } & k+1\text{-cell} \end{cases}$
- simplicial cx : 1- k-cell for each k-dim face.
- T^2 $\begin{cases} 1 \text{ of } & 0\text{-cell} \\ 2 \text{ ''} & 1 \text{ ''} \\ 1 \text{ ''} & 2 \text{ ''} \end{cases}$
- If X is a space of $\begin{cases} 1\text{- } 0\text{cell} \\ n\text{- } k\text{-cell,} \end{cases}$

  $X \cong \bigvee_{i=1}^{n} S^k$

def wedge product

let (X_i, x_i), $i \in I$ are pointed spaces, the wedge product is

$$\bigvee_{i \in I} (X_i, x_i) = \coprod X_i \Big/ \textstyle\coprod x_i$$

Projective Spaces

def. The n-dim'l projective space $\mathbb{C}P^n$

$$\mathbb{C}P^n = \mathbb{C}^{n+1} - \{0\} \Big/ \mathbb{C}^*$$

where $\mathbb{C}^*$ acts by $\lambda \vec{z} = \lambda \vec{z}$

$$\left[\begin{array}{l} Z: \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n \\ (x_0, \cdots x_n) \longmapsto [x_0 : x_1 : \cdots : x_n]. \end{array} \right.$$

Prop. $\mathbb{C}P^n$ is compact Hausdorff & $\mathbb{C}P^n \cong S^{2n+1}/S^1$

have $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$

$$\left\{ \begin{array}{l} \mathbb{C}^{n+1} - \{0\} \big/ \mathbb{R}_{>0} \cong S^{2n+1} \\ \qquad\qquad z \mapsto z/\|z\| \end{array} \right.$$

$$\mathbb{C}P^n \cong \mathbb{C}^{n+1} - \{0\} \Big/ \mathbb{R}_{>0} \times S^1 \cong S^{2n+1}/S^1 \qquad \Rightarrow \text{compact \& Hausdorff.}$$

$$\cong \underbrace{\mathbb{C}^{n+1} - \{0\}}_{S^{2n+1}} \Big/ \mathbb{C}^*$$

Def the Hopf map

$P_n : S^{2n+1} \longrightarrow \mathbb{C}P^n$ is the projection

Prop. using Hopf map to inductively construct $\mathbb{C}P^n$

$\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \cup_{P_{n-1}} D^{2n}$ where $P_{n-1} : S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$

two steps. 1st step: $\mathbb{C}P^{n-1} \cup_{P_{n-1}} D^{2n}$ glues to $\mathbb{C}P^n$

2nd step: show isomorphism

$$\mathbb{C}^n/\sim \quad \mathbb{R}^{2n}/\sim$$

Step 1:

$$\begin{array}{l} \quad \| \\ i_1 : \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n \\ \quad [\vec{z}] \longmapsto [\vec{z} : 0] \end{array}$$

$$\begin{array}{l} i_2 : D^{2n} \longrightarrow \mathbb{C}P^n \\ \quad\ \| \\ \{ z \in \mathbb{C}^n, \|z\| < 1 \} \\ \quad \vec{z} \longmapsto [\vec{z} : \sqrt{1 - \|\vec{z}\|^2}] \end{array}$$

note: $i_1|_{S^{2n-1}} = i_1 \circ P_{n-1}$ so i_1, i_2 agree on $\partial D^{2n} = S^{2n-1}$ and P_{n-1} attach S^{2n-1} to $\mathbb{C}P^{n-1}$

combining i_1, i_2, define $i: D^{2n} \cup_{P_{n-1}} \mathbb{CP}^{n-1} \to \mathbb{CP}^n$

Step 2: Check bijection.

$i: D^{2n} \cup_{P_{n-1}} \mathbb{CP}^{n-1} \to \mathbb{CP}^n$

inverse of i is given by

$$i^{-1}[z_0 : z_1 : \cdots : z_n] = \begin{cases} \text{if } z_n \neq 0 & \text{then } (z_0, z_1, \cdots, z_{n-1}) \in D^{2n} \\ \text{if } z_n = 0 & \text{then } [z_0 : z_1 : \cdots : z_{n-1}] \in \mathbb{CP}^{n-1} \end{cases}$$

Proof scheme:

↳ goal: $\mathbb{CP}^{n-1} \cup_{P_{n-1}} D^{2n} \cong \mathbb{CP}^n$

↳ $i_1 : \mathbb{CP}^{n-1} \to \mathbb{CP}^n$ $\quad i_1|_{S^{2n-1}} = P_{n-1}$ so glue

$[\vec{z}] \to [\vec{z} : 0]$

$i_2 : D^{2n} \to \mathbb{CP}^n$

$\vec{z} \to [\vec{z} : \sqrt{1 - \|z\|^2}]$

↳ reverse of i exists

Prop. The cellular construction of $\mathbb{CP}^n$

$\mathbb{CP}^n$ is a fcc with one cell of dim $2i$, $0 \leq i \leq n$, and no other cells.

Ex $\mathbb{CP}^1 \cong S^2$ is a Remann sphere.

a 0-cell and a 2-cell $\Rightarrow$ only attaching map is $S^1 \to D^0$. get S^2.

def. $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\}) / \mathbb{R}^\times \cong S^n / (\mathbb{Z}/2\mathbb{Z})$ where $\mathbb{Z}/2\mathbb{Z} : x \mapsto -x$

and similar arguments show $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_{P_{n-1}} D^n$

$\mathbb{RP}^n$ is a fcc with 1-cell of dim i, $0 \leq i \leq n$. ← show?

Prop computing $H_*(\mathbb{CP}^n)$

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2, \cdots, 2n \\ 0 & \text{o.w.} \end{cases}$$

Proof: consider LES of pair $H_*(\mathbb{CP}^n, \mathbb{CP}^{n-1})$

note that $\mathbb{CP}^n / \mathbb{CP}^{n-1} \cong S^{2n}$ where S^{2n} has $\begin{cases} 1 \text{ 0-cell} \\ 1 \text{ 2n-cell} \end{cases}$

$$H_*(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \underset{\substack{\uparrow \\ \text{collapsing} \\ \text{of a pair}}}{\cong} \tilde{H}_*(S^{2n}) = \begin{cases} \mathbb{Z} & \text{if } * = 2n \\ 0 & \text{o.w.} \end{cases}$$

using LES of pair yields

$$\begin{tikzcd}
 & & H_{i+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \arrow[dll, "\partial"'] \\
H_i(\mathbb{CP}^{n-1}) \arrow[r] & H_i(\mathbb{CP}^n) \arrow[r] & H_i(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \arrow[dll, "\partial"'] \\
H_{i-1}(\mathbb{CP}^{n-1}) \arrow[r] & {}
\end{tikzcd}$$

note that all $\partial=0$. $H_{i+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to H_i(\mathbb{CP}^{n-1})$, when $i \neq 2n-1$, domain is 0. when $i=2n-1$, $H_{2n-1}(\mathbb{CP}^{n-1})$ is 0. So ∂ always 0.

$$0 \to H_i(\mathbb{CP}^{n-1}) \to H_i(\mathbb{CP}^n) \to \underset{\overset{\uparrow}{\text{free}}}{H_i(\mathbb{CP}^n, \mathbb{CP}^{n-1})} \to 0$$

$$\Rightarrow \quad H_i(\mathbb{CP}^n) = H_i(\mathbb{CP}^{n-1}) \oplus H_i(S^{2n}) = \begin{cases} \mathbb{Z} & \text{when } i \text{ even} \\ 0 & \text{o.w.} \end{cases}$$

→ think about computing $H_i(\mathbb{RP}^n)$.

week 4 lec 2

Prop $H_k(D^k, S^{k-1}) \xrightarrow{\cong} H_{k-1}(S^{k-1})$

Proof: In the LES of (D^k, S^{k-1}), $\tilde{H}_k(D^k)=0$

define $[D^k, S^{k-1}] \xrightarrow{\cong} [S^{k-1}]$

Prop: (X_k, X_{k-1}) is a good pair

let X be a f.c.c.

let A_k = set of k cells of X

$X_k = X_{k-1} \cup_{\amalg f_\alpha} (\coprod_{\alpha \in A_k} D^k)$ $\quad f_\alpha : S^{k-1} \to X_{k-1}$

let $U_k = X_{k-1} \cup_{\amalg f_\alpha} (\coprod_{\alpha \in A_k} D^k - 0)$ ← note: still legal. just remove the center point but boundary map still sticks same.

S^{k-1} is a d.r. of $D^k - 0$

X_{k-1} is a d.r. of U_k $\Rightarrow$ X_{k-1} is a d.r. of $U_k \underset{\text{open}}{\subset} X_k$

$\Rightarrow$ (X_k, X_{k-1}) is a good pair.

we can do it because it's a good pair.

Prop. important properties

↳ $X_k/X_{k-1} \cong \bigvee_{\alpha\in A_k} S^k$

↳ $\boxed{H_k(X_k, X_{k-1})} \cong H_k\left(\bigvee_{\alpha\in A_k} S^k\right)$

$$\begin{tikzcd}[row sep=small] {} \arrow[d, phantom, "\cong" rotate=90] \\ {} \end{tikzcd}$$

$H_k\left(\coprod_{\alpha\in A_k} D^k, \coprod_{\alpha\in A_k} S^{k-1}\right) = \bigoplus_{\alpha\in A_k} H_k(D^k, S^{k-1}) = \boxed{\langle e_\alpha \mid \alpha\in A_k\rangle}$

where $e_\alpha = \Phi_{\alpha *}[D^k, S^{k-1}]$.

def. p_β map

have

$$\begin{tikzcd} \bigvee_{\alpha\in A_k} S^k \arrow[r] \arrow[dr, "p_\beta"'] & \bigvee_{\alpha\in A_k} S^k \Big/ \bigvee_{\alpha\neq\beta} S^k \arrow[d, phantom, "\cong" rotate=90] \\ & S^k \end{tikzcd}$$

$p_\beta:$ projection onto the β^{th} cell.

e.g. $p_\beta(e_\alpha) = \begin{cases} [S^k] & \alpha=\beta \\ 0 & \alpha\neq\beta. \end{cases}$

def d_k (d_k^{cell} for cellular homology)

d_k is the boundary map $d_k: H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$

In the LES of triple (X_k, X_{k-1}, X_{k-2}).

lemma $d_k = (\pi_{k-1})_* \circ \delta_k$

$d_k = (\pi_{k-1})_* \circ \delta_k$

where $\delta_k: H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1})$ is the boundary map in the LES of pair (X_k, X_{k-1})

$(\pi_{k-1})_*: H_{k-1}(X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$ is the homology induced by projection.

(or $\pi_{k-1}: (X_{k-1}, \phi) \to (X_{k-1}, X_{k-2})$ as a map of pairs)

Proof: $d_k: H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$ $\delta_k: H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1})$

??? ← Kinda confused.

$[c]\in H_k(X_k, X_{k-1})$. let $c\in C_k(X_k)$ so $dc\in C_{k-1}(X_k)$

then $\partial_k[c] = [dc] \in H_{k-1}(X_{k-1})$

$d_k[c] = [dc] \in H_{k-1}(X_{k-1}, X_{k-2})$ so $\pi_{k-1}\partial_k[c] = d_k[c]$

cor $d_k \circ d_{k+1} = 0$

Proof: $d_k\circ d_{k+1} = (\pi_{k-1})_*\circ\partial_k \circ (\pi_k)_*\circ\partial_{k+1}$

$= (\pi_{k-1})_*\left(\partial_k\circ(\pi_k)_*\right)\circ\partial_{k+1}$ since $\partial_k\circ(\pi_k)$ are 2 consec map in LES of (X_k, X_{k-1})

$H_k(X_k) \xrightarrow{\pi_k} H_k(X_k, X_{k-1}) \xrightarrow{\partial_k} H_{k-1}(X_{k-1})$

Proof scheme :

↳ $d: H(\_\_) \to H(\_\_)$ is bd map of LES of triple

↳ Write $d = \pi \circ \delta$

↳ $d \circ d = \pi \circ (\delta \circ \pi) \circ \delta$ ↗ $= 0$ in LES.

def the cellular chain complex of X

let X be a fcc. Then $C_i^{cell}(X) = H_i(X_i, X_{i-1})$ $d_\cdot^{cell}$ is bd map in LES of triple.

then the cellular chain cx of X is $(C_\cdot^{cell}(X), d_\cdot^{cell}) = (\bigoplus H_k(X_k, X_{k-1}), \bigoplus d_k)$

Thm (Big thm of $H_*^{cell}(X)$)

1) $H_*^{cell}(X) = H_*(C_*^{cell}(X)) \cong H_*(X)$

2) computing $H_*^{cell}(X)$ works as follows:

$C_k^{cell}(X) = H_k(X_k, X_{k-1}) \cong \langle e_\alpha \mid \alpha \in A_k \rangle$

$d_k^{cell} : C_k^{cell}(X) \to C_{k-1}^{cell}(X)$

$\boxed{d_k^{cell}(e_\alpha) = \sum_{\beta \in A_{k-1}} n_{\alpha\beta} e_\beta}$ where $n_{\alpha\beta} = \deg \rho_\beta \circ f_\alpha$

$$\begin{tikzcd}
S^{k-1} \arrow[r, "\tau_\alpha"] \arrow[rrd, "\rho_\beta \circ f_\alpha"'] & X_{k-1} \arrow[r, "\pi_{k-1}"] & \bigvee_{\beta \in A_{k-1}} S^{k-1} \arrow[d, "\rho_\beta"] \\
 & & S^{k-1}
\end{tikzcd}$$

$\underbrace{\tau_\alpha \quad \pi_{k-1}}_{f_\alpha}$

Proof $d_k(e_\alpha) = (\pi_{k-1})_* \circ \partial_k (\tau_{\alpha *}[D^k, S^{k-1}])$

$= (\pi_{k-1})_* \circ \tau_{\alpha *}(\partial_k [D^k, S^{k-1}])$

$= \underbrace{(\pi_{k-1})_* \circ \tau_{\alpha *}[S^{k-1}]}_{\text{include } S^{k-1} \text{ into } X_{k-1} \text{ corresponds to the attaching map}} = f_{\alpha *}[S^{k-1}]$

So $d_k(e_\alpha) = f_{\alpha *}[S^{k-1}]$

now, coefficient of e_β in $f_{\alpha *}[S^{k-1}]$

= coefficient of $[S^{k-1}]$ in $(\rho_\beta \circ f_\alpha)_*[S^{k-1}]$

= $\deg(\rho_\beta \circ f_\alpha)$

??? ???

Don't get this part!

rest of the proof is shown later.

Ex 1. Compute $H_*^{cell}(\mathbb{C}P^n)$

$\mathbb{C}P^n$ has 1 cell of dim $2i$ for each $0 \le i \le n$.

$$\begin{tikzcd} \overset{2n}{\mathbb{Z}} \arrow[r] & \overset{2n-1}{0} \arrow[r] & \overset{2n-2}{\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \overset{1}{0} \arrow[r] & \overset{0}{\mathbb{Z}} \end{tikzcd}$$

so each $d^{cell} = 0$

$\Rightarrow H_*(\mathbb{C}P^n) \cong C_*^{cell}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } * \text{ is } 0, 2, \cdots 2n \\ 0 & \text{o.w.} \end{cases}$

Ex 2. $\mathbb{R}P^n$

$\mathbb{R}P^n$ has 1 cell of dim K for each $0 \le K \le n$.

$C_K^{cell}(\mathbb{R}P^n) = \langle e_K \rangle$

$$\begin{tikzcd} \underset{\langle e_n \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \arrow[r, "d_n"] & \underset{\langle e_{n-1} \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \arrow[r, "d_{n-1}"] & \mathbb{Z} \arrow[r, "d_{n-2}"] & \cdots & \underset{\langle e_1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \arrow[r, "d_1"] & \underset{\langle e_0 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \end{tikzcd}$$

key to understand this:

- understand the map $d: C_K^{cell}(X_K) \to C_{K-1}^{cell}(X_{K-1})$
- write g_{K-1} = composition of long chain of maps
- pick q, $h_{K-1}(q) \in \mathbb{R}P^{K-1} \setminus \mathbb{R}P^{K-2}$, open nbd $\Rightarrow \tilde{g} = \pm 1$
- $h_{K-1} = h_{K-1} \circ A$ since h_{K-1} identifies antipodal points
- write as sum
- odd $\to 0$
  even $\to \pm 2$
- get result.

lemma that helps to prove $H_\bullet^{cell}(X) = H_*(X)$ $\boxed{\tilde{H}_*(X) = 0 \quad * < m, \ * > M}$

If X is a fcc with one 0 cell, all other cells have dim d, $m \le \dim \le M$

then $\tilde{H}_\bullet(X) = 0$ when $* < m$ or $* > M$

Proof: By induction on $M - m$

base case: $M - m = 0 \Rightarrow M = m$

then X has 1 cell in dim 0, all other cell with dim $m = M$.

$X \cong \bigvee_{\alpha \in A} S^n$ so $\tilde{H}_i(X) = 0$ for $i \ne m$

Inductive step:

Suppose statement holds when $M - m < K$. let $M - m = K$.

Suppose X has cell of dim $M \le \dim \le m+K$, Then X_{m+K-1} has cell between m and $m+K-1$.

inductive hypothesis apply to X_{m+K-1},

Consider LES of pair (X, X_{m+K-1}). It's a good pair. $X = X_{m+K}$, $X / X_{m+K-1} = \bigvee_{\alpha \in A} S^{m+K}$

$\Rightarrow H_*(X, X_{m+K-1}) = 0$ unless $* = m+K$

$\Rightarrow \tilde{H}_*(X_{m+K-1}) = 0$ unless $m \le * \le m+K-1$

note: $\tilde{H}_*(X_{m+K-1}) \to \tilde{H}_*(X) \to H_*(X, X_{m+K-1})$

unless $* = m+K$ or $* \in [m, m+K-1]$, both are 0. so when $* \notin [m, m+K]$,

we will have they both 0 so $\tilde{H}_*(X)$ is 0.

Proof scheme:

inductive process: $\begin{cases} H_*(X, X_{m+k-1}) = 0 & \text{unless } * = m+k \\ \tilde{H}_*(X_{m+k-1}) = 0 & \text{unless } m \le * \le m+k-1 \end{cases}$

then use $\tilde{H}_*(X_{m+k-1}) \to \tilde{H}_*(X) \to H_*(X, X_{m+k-1})$

week 4 lec 3

(Recall lemma) (all cells $m \le \dim \le M$) $\tilde{H}_*(X) = 0$ $* \notin [m, M]$.

lemma If X is a fcc, (X, X_k) is a good pair ← ??? where did you prove this?

Cor $H_k(X_{k+1}) = H_k(X)$

Proof: LES of $H(X, X_{k+1})$

$$\overset{=0}{H_{k+1}(X, X_{k+1})} \to H_k(X_{k+1}) \xrightarrow{i_*} H_k(X) \to \overset{=0}{H_k(X, X_{k+1})}$$

By collapsing a pair,

$$H_k(X, X_{k+1}) \cong \tilde{H}_k(X/X_{k+1})$$

X/X_{k+1} has one 0-cell, all other cell with dim $\geq k+2$.

So by lemma, $H_k(X/X_{k+1}) = H_{k+1}(X/X_{k+1}) = 0$

Thm. If X is a fcc, then $H_*^{cell}(X) \cong H_*(X)$.

Proof. to reconstruct the "cellular net", start with the H_*^{cell} horizontal. Then the two Δs to get $d = \pi \circ \delta$. Then, work it out.

Proof:

$$\begin{tikzcd}
 & & & & \overset{=0}{\boxed{H_k(X, X_{k+1})}} & & \\
 & \overset{=0}{\boxed{H_k(X_{k-1})}} \arrow[dr] & & H_k(X_{k+1}) \arrow[ur] & & & \\
 & & H_k(X_k) \arrow[ur, "\text{surj } i"] \arrow[dr, leftrightarrow, "\pi_k \text{ inj} \Rightarrow \cong"] & & & & \\
{} \arrow[r] & H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\delta_{k+1}"] \arrow[rr, "d_k"] & & H_k(X_k, X_{k-1}) \arrow[rr, "d_{k-1}"] \arrow[dr, "\delta_k"] & & H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r] & {} \\
 & & & & H_{k-1}(X_{k-1}) \arrow[ur, leftrightarrow, "\pi_{k-1} \text{ inj} \Rightarrow \cong"'] & & \\
 & & & \overset{=0}{\boxed{H_{k-1}(X_{k-2})}} \arrow[ur] & & &
\end{tikzcd}$$

Horizontal is chain

Diagonals are exact.

so π_{k-1}, π_k are injections, i is surjective.

$$\ker(d_{k-1}) = \ker(\pi_{k-1} \circ \delta_k) \underset{\substack{\uparrow \\ \pi_{k-1} \text{ is} \\ \text{injective}}}{=} \ker(\delta_k) \underset{\substack{\uparrow \\ \text{exactness}}}{=} \operatorname{im}(\pi_k) \underset{\substack{\uparrow \\ \pi_k \text{ bijection.}}}{\cong} H_k(X_k)$$

$$\operatorname{im}(d_k) = \operatorname{im}(\pi_k \circ \partial_{k+1}) \underset{\substack{\uparrow \\ \pi_k \text{ bijective}}}{\cong} \operatorname{im}(\partial_{k+1}) \qquad H_k^{cell}(X) = \frac{\ker(d_{k-1})}{\operatorname{im}(d_k)} = \frac{H_k(X_k)}{\operatorname{Im}\delta_{k+1}} = \frac{H_k(X_k)}{\ker(i)} \cong \operatorname{im}(i) = H_k(X_{k+1})$$

$$H_k^{cell}(X) = \frac{\ker(d_{k-1})}{\operatorname{im}(d_k)} = H_k(X_{k+1}) = H_k(X_k)$$

2.3. Homology w/ coefficients

def. tensor product

If M, N are R-modules, then

$M \otimes N$ is the R module $\langle m \otimes n \mid m \in M, n \in N \rangle / \sim$

where $\sim$ is $\begin{cases} \text{component-wise distributivity} \\ \text{scalar prod with coefficient in } R. \end{cases}$

Properties ① $M \otimes N \cong N \otimes M$ ② $R \otimes M \cong M$ ③ $(M_1 \oplus M_2) \otimes M_3 \cong M_1 \otimes M_3 \oplus M_2 \otimes M_3$.

Examples 1) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ as $q \otimes k = q/n \otimes nk = (q/n) \otimes 0 = 0$

2) $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} = \mathbb{Z}/\gcd(a,b)\mathbb{Z}$ (see proof)

def. $\otimes M$ functor

$\otimes M$ gies a functor

$$\begin{Bmatrix} R \text{ modules} \\ R \text{ linear maps} \end{Bmatrix} \xrightarrow[f \mapsto f \otimes 1]{N \mapsto N \otimes M} \begin{Bmatrix} R\text{-modules} \\ R\text{-linear mps} \end{Bmatrix}$$

$$f \otimes 1 \, (n \otimes m) = f(n) \otimes m$$

So if (C, d) is a chain cx, $(C \otimes M, d \otimes 1)$ is another chain cx.

def. singular chain cx with coefficient in G.

If G is a $\mathbb{Z}$-module / abelian grop,

$C_*(X;G) = C_*(X) \otimes G$ is singular chain cx with coefficent in G.

$H_*(X,G)$ is its homology.

Ex. chain cx over R to chain cx over R'

let R' be a ring that's also an R-module.

then there's a functor
$$\begin{Bmatrix} R\text{- mod} \\ R\text{-lin mps} \end{Bmatrix} \xrightarrow[M \mapsto M \otimes R']{\otimes R'} \begin{Bmatrix} R' \text{ modules} \\ R' \text{ linear maps} \end{Bmatrix}$$

induces a functor
$$\begin{Bmatrix} \text{chain cx over } R \\ \text{chain maps} \end{Bmatrix} \xrightarrow{\otimes R'} \begin{Bmatrix} \text{chain cx over } R' \\ \text{chain mps} \end{Bmatrix}$$

lemma: $f, g : C \to C'$ are chain homotopic via h

then $f \otimes 1, g \otimes 1 : C \otimes M \to C' \otimes M$: are homotopic via $h \otimes 1$.

Example : tensoring with R' - an R module.

Ex : $C = C_*^{cell}(\mathbb{RP}^3)$

$$\begin{array}{c|ccccccc}
C_*^{cell} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} \\
 & 3 & & 2 & & 1 & & 0 \\
\hline
H_*(C) & \mathbb{Z} & & 0 & & \mathbb{Z}/2 & & \mathbb{Z} \\
\hline
C_* \otimes \mathbb{Q} & \mathbb{Q} & \xrightarrow{\cdot 0} & \mathbb{Q} & \xrightarrow{\cdot 2} & \mathbb{Q} & \xrightarrow{\cdot 0} & \mathbb{Q} \\
\hline
H_*(C_* \otimes \mathbb{Q}) & \mathbb{Q} & & 0 & & 0 & & \mathbb{Q} & = H_*(C) \otimes \mathbb{Q} \\
\hline
C_* \otimes \mathbb{Z}/2 & \mathbb{Z}/2 & \xrightarrow{\cdot 0} & \mathbb{Z}/2 & \xrightarrow{\cdot 0} & \mathbb{Z}/2 & \xrightarrow{\cdot 0} & \mathbb{Z}/2 \\
\hline
H_*(C_* \otimes \mathbb{Z}/2) & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & \neq H_*(C) \otimes \mathbb{Z}/2 \\
\hline
\end{array}$$

def. Euler char

let C be a f.d. chain cx over a field. let $c_k = \dim(C_k)$ let $h_k = \dim(H_k)$.

then $\chi(C) = \sum_k (-1)^k c_k$

thm $\chi(C) = \chi(H_*(C)) = \sum (-1)^k h_k$

pf: let $z_k = \dim\ker(d_k)$ $b_k = \dim \operatorname{im}(d_k)$

$c_k = z_k + b_k$.

$H_k(C) = \dfrac{\ker d_k}{\operatorname{im} d_{k+1}}$ so $h_k = z_k - b_{k+1}$

$\chi(C) = \sum (-1)^k z_k + (-1)^k b_k$

$$\chi(h) = \sum (-1)^k (z_k - b_{k+1}) = \sum (-1)^k z_k - (-1)^k b_{k+1}$$
$$= \sum (-1)^k z_k + (-1)^k b_k$$

The Eilenberg Steenrod Axioms

def. an ordinary homology theory w/ coeff in G (abelian group)

is a functor $\left\{ \begin{array}{l} \text{Pair of spaces} \\ \text{mp of pairs} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \mathbb{Z} \text{ modules} \\ \mathbb{Z}\text{-liner mps} \end{array} \right\}$

$(X,A) \to H(X,A)$

$f \to f_*$

Satisfying
1) homotopy invariance
2) LES of a pair & mp of pairs induces a mp of LES.
3) Excision thm
4) Dimension axiom: $H_*\{\cdot\} = \begin{cases} G & * = 0 \\ 0 & \text{o.w.} \end{cases}$

Thm. if X is a fcc, and H_* is any functor satisfying above,

then $H_*(X) \cong H_*(C_*^{cell}(X) \otimes G) \cong H_*(X;G)$

in particular, $H_*(X;G)$ satisfy these axioms.

(means can tensor $\underbrace{\text{any}}_{\text{any fcc's hom with}}$ R', as long as R' is a R module)

Week 5 lec 1

def. free resolution

M is a M-module. Then A is a free.res. of M if A is a free chain complex s.t.

1) $A_* = 0$ for $* < 1$

2) $H_0(A) = M$

$H_*(A) = 0$ for $* \neq 0$.

e.g. if R is a PID

$$\begin{tikzcd} 3 & 2 & 1 & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot a"] & R \arrow[r] & 0 \end{tikzcd}$$

free.res of $R/(a)$

$R = \mathbb{C}[x,y]$

$$\begin{tikzcd} R \arrow[r, "{[x\ y]}"] & R^2 \arrow[r, "{\begin{bmatrix} -y \\ x \end{bmatrix}}"] & R \\ 2 & 1 & 0 \end{tikzcd}$$

free.res of $R/(x,y)$

def. $Tor_i(M,N)$ for M,N modules.

let M, N be modules. $Tor_i(M,N) = H_i(A \otimes N)$ where A is a free.res of M.

Tor measures the failure of $H_*(A \otimes N)$ to be $H_*(A) \otimes N$. (at 0 = $M \otimes N$).

Prop. $Tor_i(M,N)$ is well defined

fact: any 2 free-resolution of M are chain htp equivalent.

$A \sim A' \Rightarrow A \otimes N \sim A' \otimes N \Rightarrow H_*(A \otimes N) = H_*(A' \otimes N)$

fact $Tor_0 = M \otimes N$.

example of $Tor_*(\mathbb{Z}/a, \mathbb{Z})$ and $Tor_*(\mathbb{Z}/a, \mathbb{Z}/b)$

recall any $\mathbb{Z}$ module R, $\mathbb{Z} \otimes_{\mathbb{Z}} R = R$.

fact $Tor_*(\mathbb{Z}/2, \mathbb{Z}/2)$ explains the extra $\mathbb{Z}/2$. in $H_*(C_*^{cell}(\mathbb{R}P^2))$

defn short injective

a chain CX is short injective if

1) $C_* = 0$ for $* \neq k, k+1$, C_k, C_{k+1} is free over R

2) $d: C_{k+1} \to C_k$ is injective

Thm structure thm for chain complex over a PID.

a free chain cx over a PID is a direct sum of short injective cxs.

Proof

fact if R is a PID and M is a free module over R. Then all M's submodules are free.

actual proof

We have a SES

$$0 \to \ker d_k \to C_k \to \operatorname{Im} d_k \to 0$$

↳ im d_k is free

$\operatorname{Im} d_k \subset C_{k-1}$, C_{k-1} is free, so im d_k is free. So the chain splits.

↳ so $C_k = \ker d_k \oplus B_k$ where $B_k \overset{d_k}{\cong} \operatorname{Im} d_k$ $d^2=0 \Rightarrow \operatorname{Im} d_k \subset \ker d_{k-1}$

so we get a map $B_k \to \ker d_{k-1}$ where the map is injective

↳ each $0 \to B_k \to \ker d_{k-1} \to 0$ is a chain cx, injective

↳ $C = \bigoplus (B_k \to \ker d_{k-1})$

$$\begin{tikzcd}
 & C_k \arrow[dd] & & \\
B_{k+1} \arrow[r] & \ker d_k & C_{k-1} \arrow[dd] & \\
 & B_k \arrow[r] & \ker d_{k-1} & \\
 & & B_{k-1} \arrow[r] & \ker d_{k-2}
\end{tikzcd}$$

Cor. If two free chain cx over a PID have ≅ homology, then they're ~ equivalent.

Proof: each free chain cx is a direct sum of free resolutions of their homology. By fact that any two free resolutions of same module are hty equivalent, so is the two chains.

Cor If C is a chain cx over a field $\mathbb{F}$, then $C \overset{\text{chain hty equivalent}}{\sim} (H_*(C), 0)$

Proof: C is chain cx

$(H_*(C), 0)$ is $\cdots \to H_5(C) \xrightarrow{\cdot 0} H_4(C) \xrightarrow{\cdot 0} H_3(C) \xrightarrow{\cdot 0} \cdots$

have same homology. H is free over $\mathbb{F}$ as every module over $\mathbb{F}$ is free.

vector spaces always free

hence they're chain hty equivalent

$\text{Tor}_i(M,N) = H_i(\underset{\uparrow}{A}\otimes N)$

a fres of M.

Cor (universal coefficient thm)

let C be a free chain cx over a PID, then

$$H_k(C\otimes N) = H_k(C)\otimes N \oplus \text{Tor}_1(H_{k-1}(C), N)$$
$$= \text{Tor}_0(H_k(C), N) \oplus \text{Tor}_1(H_{k-1}(C), N)$$

Proof: C is a ict sum of short injective cx.

Suffice to^{dl} check on a short injective cx.

← ???

No idea how to show for S.I.CX.

$$0\otimes N \longrightarrow \overset{1}{C_1\otimes N} \xrightarrow{\bar{i}} \overset{0}{C_0\otimes N} \longrightarrow 0\otimes N$$

WTS $H_k(C\otimes N) = \big(H_k(C)\otimes N\big) \oplus \text{Tor}_1(\underset{\uparrow}{H_{k-1}(C)}, N)$

$C_1 \rightarrow C_0$ is a free res of $H_0(C)$.

???

$= H_{k-1}(C\otimes N)$

$k=1$, $LHS=0$ $RHS = 0 \oplus H_0(C\otimes N)$

$k=0$ $LHS = H_0(C\otimes N)$, $RHS = H_0(C)\otimes N$

as a result of universal coefficient thm, $H_*(X;G)$ is determined by $H_*(X) \oplus \text{Tor}$.

III) Cohomology & Products.

def M, N are R modules, then Hom(M,N) is R-module

def f^*

let $f: M_1 \rightarrow M_2$, then $f^*: \text{Hom}(M_2, N) \rightarrow \text{Hom}(M_1, N)$

$$\alpha \mapsto \alpha\circ f$$

f^* is R-linear

def. contravariant functor

almost same as a functor except $F(f\circ g) = F(g)\circ F(f)$

$F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$

objects: $X \rightarrow F(X)$ morphisms: $f: X\rightarrow Y \rightarrow F(f): F(Y) \rightarrow F(X)$

satisfying $F(1_X) = 1_{F(X)}$ $F(f\circ g) = F(g)\circ F(f)$

Prop: * defines a contravariant functor.

$$(f\circ g)^*\alpha = \alpha\circ(f\circ g) = f^*(\alpha)\circ g = g^*(f^*(\alpha))$$

$$\left\{\begin{matrix}\text{R-modules}\\ \text{R-liner maps}\end{matrix}\right\} \to \left\{\begin{matrix}\text{R-modules}\\ \text{R-liner maps}\end{matrix}\right\}$$

$$M \to \mathrm{Hom}(M,N)$$

$$f \mapsto f^*$$

$$(f: M_1\to M_2) \qquad (f^*: \mathrm{Hom}(M_2,N)\to \mathrm{Hom}(M_1,N))$$

def $(\mathrm{Hom}(C,N), d^*)$ cochain complex

let (C,d) be a chain cx.

then $(\mathrm{Hom}(C,N), d^*) = \oplus_k \mathrm{Hom}(C_k,N)$

$d_k^*: \mathrm{Hom}(C_{k-1},N) \to \mathrm{Hom}(C_k,N)$ satisfy $(d^*)^2=0$

this is a cochain complex.

def covariant functors = functors.

def Covaiant functors (chain cx) → (cochain cx):

$$\left\{\begin{matrix}\text{chain cx over } R\\ \text{chain maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{cochain cx}\\ \text{cochain mps}\end{matrix}\right\}$$

$$(C,d) \longrightarrow (\mathrm{Hom}(C,N), d^*)$$

$$f: (C,d)\to(C',d') \qquad f^*: (\mathrm{Hom}(C',N), d'^*) \to (\mathrm{Hom}(C,N), d)$$

def Cohomology

let (C^*, d_k^*) be cochain cx. Then

$$H^*(C) = \frac{\ker d_k^*}{\mathrm{im}\, d_{k-1}^*}$$

def. singular cochain complex with coefficients in G

let X be a top space. Then its singular cochain cx w/ coeff in G is

$$(\mathrm{Hom}(C_*(X), G), d^*)$$
$$\|$$
$$C^*(X;G)$$

Its k^{th} singular cohomology is $H^*(C^*(X;G)) = H^*(X;G)$

Prop. extend the contravariant functor for pairs of spaces

$$H^*(X;G) : \left\{ \begin{array}{l} \text{pair of spaces} \\ \text{mp of pairs} \end{array} \right\} \to \left\{ \begin{array}{l} \mathbb{Z} \text{ mods} \\ \mathbb{Z}\text{-lin mps} \end{array} \right\}$$

$$\text{Space } (X,A) \longmapsto H^*(X,A;G)$$

$$f:(X,A) \to (Y,B) \longmapsto f^*:(H^*(Y,B;G)) \to (H^*(X,A;G))$$

f^* is map on the cohomology induced by cochain mp $(f_\#)^*$

$$\left\{ \begin{array}{l} \text{pairs of spc} \\ \text{mp of prs} \end{array} \right\} \xrightarrow{\mathbb{Z}_*} \left\{ \begin{array}{l} \text{chain cx} \\ \text{chain mps} \end{array} \right\} \xrightarrow{Hom(-,G)} \left\{ \begin{array}{l} \text{cochain cx} \\ \text{cochn mp} \end{array} \right\} \to \left\{ \begin{array}{l} \mathbb{Z}\text{-mod} \\ \mathbb{Z} \text{ liner mps} \end{array} \right\}.$$

week 5 lec 2

note: $C^*(X;G)$ concretely:

$$C^*(X;G) = hom(C_*(X), G)$$

$$C^k(X;G) = hom(C_k(X), G)$$

$$= \{ \alpha : C_k(X) \to G \mid \alpha \text{ is } \mathbb{Z}\text{-linear} \}$$

$\alpha \in C^k(X;G)$ is uniquely specified by $\alpha(\sigma)$ for $\sigma : \Delta^k \to X$

note: $(d^*)^2 = 0$

$$d^*_k : C^k(X;G) \to C^{k+1}(X;G)$$

let $\alpha \in C^k(X;G)$. Then $d^*_k(\alpha) \in C^{k+1}(X;G) = hom(C_{k+1})$

$$d^*_k(\alpha)(\sigma) = \underset{\underset{C_k \to G}{\uparrow}}{\alpha} \; \underset{\underset{C_{k+1} \to C_k}{\uparrow}}{d_k} \; \underset{\underset{C_{k+1}}{\uparrow}}{\sigma}$$

$$\boxed{d^*_k(\alpha)(\sigma) = \alpha \, d_k \, \sigma}$$

$$d^*_{k+1} \circ d^*_k(\alpha) = \alpha \circ (d_{k+1} \circ d_k) = 0^* = 0$$

def cochain maps

If $f: X \to Y$, $\quad f^\# : C^k(Y;G) \to C^k(X;G)$

$f_\# : C_k(X) \to C_k(Y) \qquad hom(C_k(Y);G) \to hom(C_k(X), G)$

$$\text{by} \quad \underset{\underset{\in C^k(Y)}{\uparrow}}{f^\#(\alpha)} \, \underset{\underset{C_k(X)}{\uparrow}}{(\sigma)} = \underset{\underset{C_kY \to G}{\uparrow}}{\alpha} \; \underset{\underset{C_kX \to C_kY}{\uparrow}}{f_\#} \; \underset{\underset{C_k(X)}{\uparrow}}{(\sigma)} \quad = \alpha \circ \underbrace{(\underset{\underset{X \to Y}{\uparrow}}{f} \, \underset{\underset{\Delta^k \to X}{\uparrow}}{(\sigma)})}_{C_k(Y)}$$

$f^\bullet \alpha(\sigma) = \alpha f_\bullet(\sigma)$ and C^k eat stuff in C_k spit out G.

Prop. $f^\#$ is a cochain map

$d^*f^\# = f^\# d^*$

$d^*: C^k(X;G) \to C^{k+1}(X;G)$, $C^k(Y;G) \to C^{k+1}(Y;G)$

$f^\#: C^k(Y;G) \to C^k(X;G)$

let $\alpha \in C^k(Y;G)$ $\sigma \in C_k(Y)$

$$\begin{aligned}
\underset{\sim}{d^*}\underbrace{f^\#}(\alpha)(\sigma) &= f^\#(\alpha)(d\sigma) \\
&= f^\#(\alpha)\left(\Sigma_j (-1)^j \sigma \circ F_{I\setminus\{j\}}\right) \\
&= \alpha \circ f\left(\Sigma_j (-1)^j \sigma \circ F_{I\setminus\{j\}}\right) \\
&= \alpha\left(\Sigma_j (-1)^j f \circ \sigma \circ F_{I\setminus\{j\}}\right) \\
&= \Sigma_j (-1)^j \alpha \circ f \circ \sigma \circ F_{I\setminus\{j\}}) \\
&= \Sigma_j (-1)^j \alpha \circ f_\#(\sigma \circ F_j)) \\
&= \Sigma_j (-1)^j f^\# \circ \alpha(\sigma \circ F_j)) \\
&= f^\#\left(\Sigma (-1)^j \cdot (\alpha \circ \sigma) \circ F_j\right) \\
&= f^\# d(\alpha(\sigma))
\end{aligned}$$

???

the proof in note don't work?

Cor. Since $f^\#$ is a cochain map, $f^\#$ induces $f^*: H^k(Y;G) \to H^k(X;G)$

$f^*([\alpha]) \mapsto [f^\#(\alpha)]$

def. Cochain homotopies

let C, C' be cochains, $f, g: C \to C'$, are cochain maps.

then f, g are cochain homotopic if $f - g = d^*h + hd^*$ for some $h: C^k \to (C')^{k-1}$ is R-linear

lemma: If $f \sim g$ then $f^* = g^*$

lemma: If $f, g: C \to C'$ (just chain, not cochains)

$f \sim g$ via h, then $f^*, g^*: \text{Hom}(C';N) \to \text{Hom}(C;N)$ via h^*

note: things true for hom is true for coho.

Eilenberg Steenrod:

$H^*(\_\,;G)$ defines contravariant functors:

$$\left\{\begin{matrix}\text{pairs of spaces} \\ \text{mp of pairs}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{- Mod} \\ \mathbb{Z}\text{- lin mp}\end{matrix}\right\}$$

$C^*(X, A; G) \to \{f: C_k(X) \to G \mid f \text{ is } \mathbb{Z}\text{-lin}, f(\sigma) = 0 \text{ if } \text{Im}(\sigma) \subset A\}$

Prop. Properties of cohomology

1) if $f_0, f_1 : (X,A) \to (Y,B)$, $f_1 \sim f_0$, as mp of pairs, then

$$f_0^* = f_1^* : H^*(Y,A) \cong H^*(X,A)$$

Proof: $f_{0\#}, f_{1\#}$ are chain htp, so $f_0^\#, f_1^\#$ are cochain hty.

2) LES of pair

functions vanish on simplices in A

$$0 \to \overset{\downarrow}{C^*(X,A)} \longrightarrow C^*(X) \to C^*(A) \to 0$$

get LES $\to H^k(X,A) \xrightarrow{\pi^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X,A) \to \cdots$

3) excision: $B \subset A \subset X$, $\bar{B} \subset \text{int}(A)$, then

$$i^* : H^*(X,A) \to H^*(X-B, A-B) \text{ is an } \cong$$

Proof require ex shaf

4) dimension $H^*(1 \cdot 4, G) = \begin{cases} G & * = 0 \\ 0 & \text{o.w.} \end{cases}$??? how to show ?

note: work b/c homology over PID $\mathbb{Z}$, free $\Rightarrow$ $\sim$ equivalent.

Thm. Any functor satisfying above axioms for $\left\{ \begin{array}{c} \text{pair space} \\ \text{mp. pairs} \end{array} \right\} \to \left\{ \begin{array}{c} \mathbb{Z}\text{-mod} \\ \mathbb{Z}\text{-linmps.} \end{array} \right\}$

is given by $H^*_{cell}(X;G)$

where $C^*_{cell}(X;G) = \text{Hom}(C_*^{cell}(X);G)$

$$H^*_{cell}(X;G) = H^*_{cell}(C^*_{cell}(X;G))$$

Thm: $H^*_{cell}(X;G) = H^*(X;G)$ if X is a fcc

Ex. $H^*_{cell}(\mathbb{RP}^3, \mathbb{Z}/2)$

$$C_*^{cell}(\mathbb{RP}^3) = \mathbb{Z} \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}$$

$$C^*_{cell}(\mathbb{RP}^3) = \mathbb{Z} \leftarrow \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow \mathbb{Z}$$

$$H^*_{cell}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0,3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{o.w.} \end{cases}$$

Ext and UCT

def. $\text{Ext}^i(M,N)$

M, N are R-modules. Then $\text{Ext}^i(M,N) = H^i(A,N)$, A is a free res of M.

$\text{Tor}_i(M,N) = H_i(A \otimes N)$ note $\text{Ext}^0(H_k(X);G) = H^0(A;G)$

$= \text{Hom}(H_k(X);G)$

Ex $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z})$.

$A: \mathbb{Z} \xrightarrow{n} \mathbb{Z}$.

$\hom(A, \mathbb{Z}) : \underset{1}{\mathbb{Z}} \xleftarrow{\cdot n} \underset{0}{\mathbb{Z}}$ $\qquad \mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) = H^1(A, N) = \mathbb{Z}/n$.

$\mathrm{Ext}^0(\mathbb{Z}/n, \mathbb{Z}) = 0$

$\mathrm{Ext}^*(\mathbb{Z}/n, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & * = 0, 1 \\ 0 & \text{o.w.} \end{cases}$ as $H^0(A, \mathbb{Z}/n) = \mathbb{Z}/n$ by the prop above. (4 Eilenberg Steenrod ax).

Thm write H^i as Something & Ext

note we could write $H_i(A \otimes N) = H_i(A) \otimes N \oplus \mathrm{Tor}$

we also write $H^k(X; G) = \mathrm{Hom}(H_k(X); G) \oplus \mathrm{Ext}^1(H_{k-1}(X); G)$

$= \mathrm{Ext}^0(H_k(X); G) \oplus \mathrm{Ext}^1(H_{k-1}(X); G)$

Proof split $C_*^{cell}(X)$ into $\oplus$ of short injective cxs. ???

Ex. let X be a fcc

$H_k(X) = \mathbb{Z}^{b_k} \oplus T_k$ (structure thm, $\mathbb{Z}^{b_k}$ free, T_k finite)

$H^k(X) = \mathbb{Z}^{b_k} \oplus T_{k-1}$??? does T_k link for to Ext?

Pairing

def $\langle \_, \_ \rangle$ bilinear pairing

let C be a cx over R then we get bilinear pairing

$\langle \_, \_ \rangle : \mathrm{Hom}(C_k; N) \times C_k \to N$

$\langle \alpha, c \rangle \mapsto \alpha(c)$

lemma: the above pairing descends to a pairing $H^k \times H_k$

$H^k(\mathrm{Hom}(C, N)) \times H_k(C) \longrightarrow N$

$\langle [\alpha], [c] \rangle \longmapsto \alpha(c)$

Pf WTS well defined. i.e.

$\langle [\alpha + d^*\beta], [c + db] \rangle = \langle [\alpha], [c] \rangle$.

$\Rightarrow (\alpha + d^*\beta)(c + db)$

$= \alpha(c) + \alpha db + d^*\beta c + d^*\beta db$

$= \alpha(c) + d^*(\alpha b) + d^*(\beta c + \beta db)$

$= \alpha(c) + d^*(\alpha b) + \beta(dc + ddb)$

$d^*\alpha = 0$, $\Leftarrow$ α cocycle

$d(c + db) = 0$

Week 5 lec 3

cup product. (R is a commutative ring $(\mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \mathbb{R})$)

def cup product

If $\alpha \in C^k(X;R)$ $\beta \in C^l(X;R)$ then $\alpha \cup \beta \in C^{k+l}(X;R)$ given by

$$\underset{\substack{\uparrow \\ \Delta^{k+l} \to X}}{\alpha\cup\beta(\sigma)} = \alpha(\underbrace{\underbrace{\sigma}_{\Delta^{k+l}\to X} \circ \underbrace{F_{0\cdots k}}_{\Delta^k \to \Delta^{k+l}}}_{\Delta^k \to X}) \, \beta(\sigma \circ F_{k \cdots k+l})$$

where $F_{0\cdots k} : \Delta^k \to \Delta^{k+l}$

$(x_0, \cdots x_k) \to (x_0, \cdots x_k, 0, \cdots 0)$

$F_{k\cdots k+l} : \Delta^l \to \Delta^{k+l}$

$(x_0, \cdots x_l) \to (0, \cdots 0, x_0, \cdots x_l)$

lemma $\cup$ makes $C^*(X;R)$ into a commutative ring

with identity $1 \in C^0(X;R)$, $1(\sigma_p) = 1 \in R$, $\sigma_p : \Delta^0 \to X$ $(1) \to p$.

Proof check

1) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$

2) $(\alpha_1 + \alpha_2)\beta = \alpha_1 \cup \beta + \alpha_2 \cup \beta$

3) $\alpha \cup (\beta_1 + \beta_2) = \alpha \cup \beta_1 + \alpha \cup \beta_2$

4) $\alpha \cup 1 = 1 \cup \alpha = \alpha$

Verify this

lemma Leibniz rule

If $\alpha \in C^k(X;R)$ $\beta \in C^l(X;R)$ then $d^*(\alpha \cup \beta) = (d^*\alpha) \cup \beta + (-1)^k \alpha \cup (d^*\beta)$

Proof note that $\alpha \cup \beta = C^{l+k}$, $d(\alpha\cup\beta) = C^{l+k+1}$ $\sigma \in C_{l+k+1}$.

$d^*(\alpha \cup \beta)(\sigma) = (\alpha \cup \beta)\, d\sigma$

$= (\alpha\cup\beta) \sum_{j=0}^{k+l+1} (-1)^j \cdot \sigma \circ F_{\hat{j}}$

$= \sum_{j=0}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{\hat{j}} \circ F_{0\cdots k}) \, \beta(\sigma \circ F_{\hat{j}} \circ F_{k\cdots\cdots k+l})$

bump up index by 1.

Why repeat indices?

$= \sum_{j=0}^{k+1} (-1)^j \alpha(\sigma \circ F_{0\cdots\hat{j}\cdots k}) \, \beta(\sigma \circ F_{k+1\cdots k+l+1})$

$+ \sum_{j=k}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{0\cdots k}) \, \beta(\sigma \circ F_{k\cdots\hat{j}\cdots k+l+1})$

$= (d\alpha) \cup \beta + (-1)^k \alpha \cup (d\beta)$

cor $\cup$ descends to a map

$\cup \cdot H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R)$

$[\alpha] \times [\beta] \longmapsto [\alpha\cup\beta]$

this makes $H^*(X;R)$ into a ring with $[1] = 1$.

Proof Check containment then well-defined-ness.

① check containment

let $[\alpha] \in H^k(X;R)$, $[\beta] \in H^l[X;R]$

have $d^*\alpha = 0$ and $d^*\beta = 0$

then, $d^*(\alpha\cup\beta) = d^*(\alpha)\cup\beta + (-1)^l \alpha\cup d^*\beta = 0\cup\beta + (-1)^l \alpha\cup 0 = 0$

so, $[\alpha\cup\beta] \in H^{k+l}(X;R)$

② check doesn't depend on representatives

if $[\alpha'] = [\alpha]$, $\alpha' = \alpha + d^*a$

$[\beta'] = [\beta]$, $\beta' = \beta + d^*\beta$

then $\alpha'\cup\beta' = \alpha\cup\beta + (d^*a)\cup\beta + \alpha\cup d^*\beta + d^*\alpha\cup d^*\beta$

$= \alpha\cup\beta + d^*(a\cup\beta) + (\alpha + d^*a)\cup(d^*\beta)$

$\downarrow$

$d^*(a\cup\beta) = (d^*a)\cup\beta + \overbrace{(-1)^k \alpha\cup(d^*\beta)}^{=0}$

so $[\alpha\cup\beta] = [\alpha'\cup\beta']$

③ $d^* 1 = 0$. $d^* 1(z) = 1\cdot d(z) = 1_{\sigma_{z(1)}} - \sigma_{z(0)} = 1 - 1 = 0$ $z \in C_1(X)$.

Prop Continuous maps induce ring homomorphism

if $f: X \to Y$, then $f^*: H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism.

i.e. $f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta)$

Proof: 1st show # works.

consider $f^\#: C^*(Y;R) \to C^*(X;R)$

$f^\# \overset{C^{k+l}(Y)}{(\alpha\cup\beta)} \overset{C_{k+l}(X)}{(\sigma)}$

$= \overset{C^{k+l}(Y)}{(\alpha\cup\beta)} \overset{C_{k+l}(Y)}{f\circ} \overset{C_{k+l}(X)}{\sigma}$

$= \alpha\circ(f\circ\sigma\circ F_{0\cdots k})\ \beta(f\circ\sigma\circ F_{k\cdots k+l})$

$= f^\#(\alpha)\circ\sigma\circ F_{0\cdots k} \cdot f^\#(\beta)\circ\sigma\circ F_{k\cdots k+l}$

$= (f^\#(\alpha)\cup f^\#(\beta))\ \sigma$

so $f^\#(\alpha\cup\beta) = (f^\#\alpha)\cup f^\#(\beta)$

so $f^*[\alpha\cup\beta] = [f^\#(\alpha\cup\beta)] = [f^\#(\alpha)\cup f^\#(\beta)] = f^*[\alpha]\cup f^*[\beta]$

Proof scheme:

Show work for $f^\#(\alpha\cup\beta)$

use $f^*(\alpha\cup\beta) = [f^\#(\alpha\cup\beta)]$.

Prop. $\cup$ on $H^*(X;R)$ is graded commutative ~~[false for chains]~~

i.e. $a \cup \beta = (-1)^{|a||b|} \beta \cup \alpha$ $|a| = k$ if $a \in H^k(X;R)$

Proof Consider chain map $r : C_*(X) \to C_*(X)$:

$\rho_n : \Delta^n \to \Delta^n$ be liner map $\rho_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$e_i \to e_{n-i}$

let $\varepsilon(j) = \frac{j(j+1)}{2} = \sum_{i=0}^{j} i$ $\det \rho_j = (-1)^{\varepsilon_j}$

define $r_j : C_j(X) \to C_j(X)$

$\sigma \mapsto (-1)^{\varepsilon(j)} \sigma \circ \rho_j$

theorem proven later } then, 1) $r : C_*(X) \to C_*(X)$ is a chain map

2) $r \sim 1_{C_*(X)}$

now, back to proving the Proposition,

$r : C^*(X;R) \to C^*(X;R)$

dualizing $r \Rightarrow r^* : C^*(X;R) \to C^*(X;R)$ $r^* \sim 1_{C^*(X)}$ so $[r^*(\alpha)] = [\alpha]$.

then $(-1)^{\varepsilon(|\alpha|+|\beta|)} r^*(\alpha \cup \beta)$

$= (-1)^{\varepsilon|\alpha|} (-1)^{\varepsilon|\beta|} r^*(\beta) \cup r^*(\alpha)$

???

"r reverses the ordr of vertices"

i.e. $[\alpha \cup \beta] = [r^*(\alpha \cup \beta)] = (-1)^{\varepsilon(|\alpha|+|\beta|)} (-1)^{\varepsilon(|\alpha|)} (-1)^{\varepsilon(|\beta|)} [r^*(\beta) \cup r^*(\alpha)]$???

$= (-1)^{|\alpha||\beta|} [\beta] \cup [\alpha]$

Proof r is a chain map

$\rho_n \circ F_{\hat{j}} = F_{\widehat{n-j}} \circ \rho_{n-1}$ recall $r_j(\sigma) = (-1)^{\varepsilon(j)} \sigma \circ \rho_{|\sigma|}$

so, $d(r(\sigma)) = (-1)^{\varepsilon(|\sigma|)} \sum (-1)^j \sigma \circ \rho_{|\sigma|} \circ F_{\hat{j}}$

$= (-1)^{\varepsilon(|\sigma|)} \sum (-1)^j \circ \sigma \circ F_{\widehat{n-j}} \circ \rho_{|\sigma|-1}$

$= (-1)^{|\sigma|} (-1)^{\varepsilon(|\sigma|)} \sum (-1)^{n-j} \sigma \circ F_{\widehat{n-j}} \rho_{n-1}$

$= r_{n-1}(d\sigma)$

▨

Week 6 lec 1.

thm $r \sim 1_{C_*(X)}$

Proof is quite complicated here skipped. Must comeback in future!!

Come back!

Once this theorem is shown, graded commutativity is proven. Hence $H^*(X;R)$ is a graded commutative ring.

Pairs using $\mathbb{Z}$ coefficients

Recall that, $C^*(X,A) \subset C^*(X)$, $C^*(X,A) = \{ \alpha \in \hom(C_*, R) : \alpha(\sigma)=0 \text{ if } \operatorname{im}(\sigma) \subset A \}$

Prop. If $\alpha \in C^k(X,A)$, $\beta \in C^l(X)$ then $\alpha \cup \beta \in C^*(X,A)$

let $\alpha \in C^*(X,A)$, $\beta \in C^l(X)$ if $\operatorname{im}(\sigma) \subset A$, then $\operatorname{im}(\sigma \circ F_{0\cdots k}) \subset A$

$$\alpha \cup \beta(\sigma) = \alpha(\sigma \circ F_{0\cdots k}) \cdot \beta(\sigma \circ F_{k \cdots k+l})$$

$$= 0 \cdot \beta(\sigma \circ F_{k\cdots k+l}) = 0$$

so $\alpha \cup \beta \in C^*(X,A)$.

Cor. $\cup$ descends to a map $H^*(X,A) \times H^*(X) \to H^*(X,A)$

$$(\alpha, \beta) \to \alpha \cup \beta.$$

Cor. generally, $\cup$ defines a map $H^*(X,A) \times H^*(X,B) \to H^*(X, A \cup B)$

proof: example sheet.

Examples of cup products & cohomology

1) If X is path connected, $H_0(X) \cong \mathbb{Z}$. $H^0(X) \cong \hom(H_0(X), \mathbb{Z}) = \mathbb{Z}$ (since $H_{-1}(X)=0$ by UCT)

free

Recall $H^k(X;G) = \operatorname{Hom}(H_k(X);G) \oplus \operatorname{Ext}^1(H_{k-1}(X);G)$ $\quad \underset{0}{\shortparallel}$

$H^0(X) = \langle 1 \rangle$. Since if $\sigma_p \in C_0(X)$, $\langle 1, [\sigma_p] \rangle = 1$ so 1 is primitive ← not a multiple of anything other than $1, -1$.

↑ identity element in C^*.

2) Recall $H_*(S^n) = \begin{cases} \mathbb{Z} & *=0,n \\ 0 & \text{o.w.} \end{cases}$ is free over $\mathbb{Z}$.

then UCT implies

$$H^k(X,G) = \operatorname{Hom}(H_k(X);G) \oplus \underbrace{\operatorname{Ext}^1(H_{k-1}(X);G)}_{=0}$$

$$= \operatorname{Hom}(H_k(X);\mathbb{Z}) = \begin{cases} \mathbb{Z} & x=0,n \\ 0 & \text{o.w} \end{cases}$$

$\left.\begin{array}{l} H^0(S^n) = \langle 1 \rangle \\ H^n(S^n) = \langle a \rangle. \end{array}\right\}$ then $\begin{array}{l} 1 \cup 1 = 1 \\ a \cup 1 = a = 1 \cup a \\ a \cup a \in H^{2n}(S^n) = 0 \end{array}$

therefore, $H^*(S^n) = \mathbb{Z}[a] / \underbrace{(a^2)}_{a^2=0}$ $\quad |a| = n.$ grading ?

this is a rng satisfying : generated by $1, a$, $a^2=0$,

3) if X is p.c. , $p \in X$

$i_* : H_0(p) \cong H_0(X)$

$$\Rightarrow \begin{tikzcd} H^0(X) \arrow[r] \arrow[d, phantom, "\parallel"] & H^0(P) \arrow[d, phantom, "\parallel"] \\ \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

is an $\cong$. so $H^*(X,P) = \ker \left(H^*(X) \to H^*(p)\right)$

$$= \underbrace{\bigoplus_{i \geq 0} H^i(X)}_{}$$

note: this is a ring homomorphism

$\frac{H^*(X)}{\ker} \cong H^*(p)$

so $\ker \cong \frac{H^*(X)}{H^*(p)} = H^*(X,p)$

at 0, $H^0(X) \to H^0(p)$ $\ker = 0$.
at $i \neq 0$ $H^i(p) = 0$, everything in kernel.

so $H^*(X,p) \to H^*(X) \to H^*(p)$ are indued by map of pairs.
this is true as a ring homomorphism.

4) Prop. Structure of $H^*(X \amalg Y) \cong H^*(X) \times H^*(Y)$ (direct product of rings

$\cong H^*(X) \oplus H^*(Y)$

$$\underset{\in H^*(X)}{(a_1, b_1)} \cup \underset{\in H^*(Y)}{(a_2, b_2)} = (a_1 \cup a_2 , b_1 \cup b_2)$$

Proof :

show $C^*(X \amalg Y) = C^*(X) \times C^*(Y)$

since $C_*(X \amalg Y) = C_*(X) \oplus C_*(Y)$

have $C^*(X \amalg Y) = \mathrm{Hom}(C_*(X \amalg Y), \mathbb{Z})$

(α) $= \mathrm{Hom}(C_*(X) \oplus C_*(Y), \mathbb{Z})$

$= \mathrm{Hom}(C_*(X), \mathbb{Z}) \times \mathrm{Hom}(C_*(Y), \mathbb{Z})$

$= C^*(X) \times C^*(Y)$

(α_1, α_2)

define α as $\alpha(\sigma) = \begin{cases} \alpha_1(\sigma) & \text{if } \mathrm{im}\, \sigma \subset X \\ \alpha_2(\sigma) & \text{if } \mathrm{im}\, \sigma \subset Y \end{cases}$

$d^*(\alpha_1, \alpha_2) = (d^* \alpha_1, d^* \alpha_2)$ $\Rightarrow$ $H^*(X \amalg Y) \cong H^*(X) \oplus H^*(Y)$

now show $\underset{C^*(X)\ C^*(Y)}{(\alpha_1, \alpha_2)} \cup \underset{C^*(X)\ C^*(Y)}{(\beta_1, \beta_2)} = (\alpha_1 \cup \beta_1, \alpha_2 \cup \beta_2)$

See on simplices

$$\left((\alpha_1, \alpha_2) \cup (\beta_1, \beta_2)\right)(\sigma) = (\alpha_1, \alpha_2)(\sigma \circ F_{0 \cdots k}) \; (\beta_1, \beta_2)(\sigma \circ F_{k \cdots k+l})$$

$$= \begin{cases} \text{if } \mathrm{im}(\sigma) \subset X, & \alpha_1(\sigma \circ F_{0 \cdots k}) \beta_1(\sigma \circ F_{k \cdots k+l}) \\ \text{if } \mathrm{im}(\sigma) \subset Y, & \alpha_2(\sigma \circ F_{0 \cdots k}) \beta_2(\sigma \circ F_{k \cdots k+l}) \end{cases}$$

$$= (\alpha_1 \cup \beta_1 , \alpha_2 \cup \beta_2)(\sigma)$$

Proof Scheme :

Statement: $H^*(X \amalg Y) \cong H^*(X) \times H^*(Y)$

Proof : ↳ $C^*(X \amalg Y) \cong C^*(X) \times C^*(Y)$ since d^* lies in one of X,Y, makes sense w. d^*.

↳ $d^*(\alpha, \beta) = (d^*\alpha, d^*\beta) \Rightarrow H^*(\quad) \cong H^*(\,) \times H^*(\,)$

↳ $(\alpha_1, \alpha_2) \cup (\beta_1, \beta_2) = (\alpha_1 \cup \beta_1, \alpha_2 \cup \beta_2)$

Week 6 lecture 2

Recall that if X is p.c. $H^*(X,p) = \bigoplus_{i>0} H^i(X)$ is an ideal in $H^*(X)$.

Example #5

5. compute $H^*((X,p) \vee (Y,q))$

suppose $(X, p_x), (Y, p_y)$ are good pairs, and X,Y are P.C.

$(X \vee Y, p) = \pi((X \amalg Y, p_x \amalg p_y))$

collapsing a pair, $\pi^* : H^*(X \vee Y, p) \xrightarrow{\cong} H^*(X \amalg Y, p_x \amalg p_y)$ Recall π^* goes opposite direction

$= H^*(X,p) \oplus H^*(Y,p) \subset H^*(X) \oplus H^*(Y)$

So, $H^i(X \vee Y) = \begin{cases} H^i(X) \oplus H^i(Y) & i>0 \\ \langle 1 \rangle \cong \mathbb{Z} & i=0 \end{cases}$

for multiplication, $(a_1, a_2) \cup (b_1, b_2) = (a_1 \cup b_1, a_2 \cup b_2)$ if grading of $a_i, b_i > 0$

Example of cohom of wedge

$H^*(S^2 \vee S^2 \vee S^4) = \langle 1, a, a', b \rangle$, write $H^n(S^n) = \langle a_n \rangle$.

one copy of $\mathbb{Z}$ at H^2

have a, a', b defined as follows:

$a = (a_2, 0, 0) \in \overset{\mathbb{Z}}{H^2(S^2)} \oplus \overset{\mathbb{Z}}{H^2(S^2)} \oplus \overset{0}{H^2(S^4)} \cong \mathbb{Z}^2 \in H^2(S_2 \vee S_2 \vee S_4)$

$a' = (0, a_2, 0)$ " " "

$b = (0, 0, a_4) \in \overset{0}{H^4(S^2)} \oplus \overset{0}{H^4(S^2)} \oplus \overset{\mathbb{Z}}{H^4(S^4)} \cong \mathbb{Z} \in H^4(S_2 \vee S_2 \vee S_4)$

$a \cup a' = (a_2,0,0)(0,a_2,0) = (0,0,0) = 0$ no intersecting cup products.

Exterior Product.

defn. Setup for projection

Setup: (X,A) pair of spaces, Y is a space.

$$\pi_1 : (X\times Y, A\times Y) \to (X,A) \qquad \pi_2 : X\times Y \to Y$$

$$((x,y_1),(a,y_2)) \mapsto (x,a) \qquad (x,y) \mapsto y$$

def. Exterior Product.

If $a \in H^k(X,A)$, $b \in H^\ell(Y)$, then their exterior product is

$$a\times b = \underset{\underset{H^k(X\times Y, A\times Y)}{\rotatebox{90}{$\in$}}}{\pi_1^*(a)} \cup \underset{\underset{H^\ell(X\times Y)}{\rotatebox{90}{$\in$}}}{\pi_2^*(b)} \quad \underset{\uparrow}{\in} H^{k+\ell}(X\times Y, A\times Y)$$

legal as proven before.

exterior product depends on the π_1, π_2 quotient map.

observations of exterior product.

1) $H^*(X,A) \times H^*(Y) \longrightarrow H^*(X\times Y, A\times Y)$

$(a,b) \longmapsto a\times b$ is bilinear hence it extends to

$\Phi : H^*(X,A) \otimes H^*(Y) \longrightarrow H^*(X\times Y, A\times Y)$

$a\otimes b \longmapsto a\times b$

2) $(a_1\times b_1) \cup (a_2\times b_2) = (-1)^{|b_1||a_2|}(a_1\cup a_2)\times(b_1\cup b_2)$

Proof:

swap

$$\text{LHS} = (\pi_1^*(a_1)\cup\pi_2^*(b_1)) \cup (\pi_1^*(a_2)\cup\pi_2^*(b_2))$$

$$= (-1)^{|b_1||a_2|}\,\pi_1^*(a_1)\cup\pi_1^*(a_2)\cup\pi_2^*(b_1)\cup\pi_2^*(b_2)$$

$$= (-1)^{|b_1||a_2|}\,\pi_1^*(a_1\cup a_2)\cup\pi_2^*(b_1\cup b_2)$$

π^* is a ring hom w.r.t. the multiplication $\cup$.

$$= (-1)^{|b_1||a_2|}\,(a_1\cup a_2)\times(b_1\cup b_2)$$

Thm. exterior product induced tensor gives isomorphism.

If $H^*(Y;R)$ is free over R, important (i.e. if R is a field, it's free) then

$$\Phi : H^*(X,A;R) \otimes \underbrace{H^*(Y;R)}_{\text{free.}} \longrightarrow H^*(X\times Y, A\times Y;R) \quad \text{is an } \cong.$$

Consequences:

1) we can use this to compute $H^*(X\times Y;R)$ from $H^*(X;R)$, $H^*(Y;R)$

2) gives us ring structure on $H^*(X\times Y;R)$, by observation 2).

Example 1. Exterior product

$T^2 = S^1 \times S^1$ The theorem applies as $H^*(S^1)$ is free over $\mathbb{Z}$.

dim				
1	$\mathbb{Z}$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	
0	$\mathbb{Z}$	$\mathbb{Z}^0$	$\mathbb{Z}^1$	
$H^*(S)$		$\mathbb{Z}$	$\mathbb{Z}$	
	$H^*(S)$	0	1	dim

It has rank 4.

$$H^*(S^1 \times S^1) \begin{cases} \mathbb{Z} & * = 2 \quad \langle \overset{c}{\overbrace{a_1 \times a_1}} \rangle = \langle c \rangle \text{, where } \langle a_1 \rangle = H^1(S^1) \\ \mathbb{Z}^2 & * = 1 \quad \langle \overset{a}{\overbrace{a_1 \times 1}}, \overset{b}{\overbrace{1 \times a_1}} \rangle = \langle a, b \rangle . \\ \mathbb{Z} & * = 0 \quad \langle 1 \times 1 \rangle = \langle 1 \rangle \end{cases}$$

(or $\langle [S^1] \times [S^1] \rangle$

$\langle [S^1] \times 1, 1 \times [S^1] \rangle$

$\langle 1 \times 1 \rangle$.)

$a^2 = (a_1 \times 1) \cup (a_1 \times 1)$

$= (a_1^2 \times 1) = 0$

$(a_1 \times 1) \cup (a_1 \times 1)$

$(-1)^{|0||1|} (a_1 \cup a_1, 1)$

since $a_1^2 \in H^2(S^1) = 0$

$b^2 = 0$ for similar reasons.

$a \cup b = (a_1 \times 1) \cup (1 \times a_1) = (-1)^{|0||1|} (a_1 \cup 1) \times (1 \cup a_1) = a_1 \times a_1 = c$

$b \cup a = (-1)^{1 \cdot 1} a \cup b = -c$ This gives us the ring structure of $H^*(T^2)$ $\begin{cases} \{0, a, b, c\} \\ \{a^2 = b^2 = 0, \; ab = c, \; ba = -c \end{cases}$

Example. $H^*(T^n)$ as a wedge product

$H^*(T^2) = \Lambda^*(\alpha_1, \alpha_2)$, $\alpha_1 = a$, $\alpha_2 = b$,

$\alpha_i \alpha_j = -\alpha_j \alpha_i \quad \forall i, j$

More generally, $H^*(T^n) = \underbrace{H^*(S^1) \otimes \cdots \otimes H^*(S^1)}_{n \text{ times}} \trianglelefteq \Lambda^* \langle \alpha_1, \cdots \alpha_n \rangle$, $\alpha_i = 1 \times 1 \times \cdots \times 1 \times \underset{\substack{\uparrow \\ \text{position } i}}{a_1} \times 1 \times \cdots \times 1$

Ex 2. Group structure of $H^*(S^2 \times S^2)$

$H_*(S^2)$ is free so $H^*(S^2 \times S^2) = H^*(S^2) \otimes H^*(S^2)$

$\mathbb{Z}$	$\mathbb{Z}^2$		$\mathbb{Z}^4$
$\mathbb{Z}$	$\mathbb{Z}^0$		$\mathbb{Z}^2$
	$\mathbb{Z}$		$\mathbb{Z}$

$a_2 = H^2(S^2)$

$$= \begin{cases} \langle \overset{c}{\overbrace{a_2 \times a_2}} \rangle & * = 4 \\ \langle \overset{a}{\overbrace{a_2 \times 1}}, \overset{b}{\overbrace{1 \times a_2}} \rangle & * = 2 \\ \langle 1 \times 1 \rangle & * = 0 \end{cases}$$

Again, have $a \cup b = c$

But $H^*(S^2 \times S^2) \not\cong H^*(S^1 \times S^1)$ as

$b \cup a = (-1)^{2 \cdot 2} \cdot a \cup b = a \cup b = c$.

So, the structure look like $(a+b)^2 = a^2 + b^2 + 2ab = 2c$

so $\begin{cases} a^2 = 0, \; b^2 = 0 \\ \boxed{(a+b)^2 = 2c} \end{cases}$ $H^*(S^2) = 0 \quad * \neq 0, 2$.

yet for any $\alpha \in H^1(T^2)$, $\boxed{\alpha^2 = 0}$ as $\alpha \cup \alpha = -\alpha \cup \alpha$.

hence $H^*(S^1 \times S^1)$, $H^*(S^2 \times S^2)$ are different.

Cor. $S^2\times S^2$ is not hom. equivalent to $S^2\vee S^2\vee S^4$ though they have same homology.

$$H_i = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=2 \\ \mathbb{Z} & i=4 \\ 0 & o.w. \end{cases}$$

but in $H^i(S^2\times S^2)$ cup prod: $a\cup b=c$

in $H^i(S^2\vee S^2\vee S^4)$ if $|a|=|b|=2$, $a\cup b=0$.

Proof of the big thm.

If $H^*(Y;R)$ is free then,

$$\Phi: H^*(X,A;R)\otimes \underbrace{H^*(Y;R)}_{} \longrightarrow H^*(X\times Y, A\times Y;R) \quad \text{is an } \cong.$$

Proof: We have 2 contravariant functors:

$\bar{h}, \underline{h}$: $\begin{Bmatrix} \text{Pairs of spaces} \\ \text{mp of pairs} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{graded } \mathbb{Z}\text{-modules} \\ \text{graded } \mathbb{Z}\text{-linear maps} \end{Bmatrix}$

$\bar{h}(X,A) = H^*(X\times Y, A\times Y)$

$f:(X,A)\to(X',A')$ $\quad \bar{f}^*: H^*(X'\times Y, A'\times Y)\to H^*(X\times Y, A\times Y)$

$\bar{f}^* = (f\times \mathrm{Id}_Y)^*$

$\underline{h}(X,A) = H^*(X,A)\otimes H^*(Y)$

$f:(X,A)\to(X',A') \to \underline{f}^* = f^*\otimes \mathrm{Id}_Y$

$\bar{h}, \underline{h}$ Satisfy all Eilenberg Steenrod axioms for cohomology except dimension axiom.

$\Rightarrow$ they're generalized cohomology.

The axioms:

1) homotopy invariance: $f_0\sim f_1 \Rightarrow \underline{f_0}^* = \underline{f_1}^*$ $(f_0^*=f_1^*)$???

$\bar{f}_0^* = \bar{f}_1^*$ $(f_0\times 1_Y \sim f_1\times 1_Y \Rightarrow (f_0\times 1_Y)^* \sim (f_1\times 1_Y)^*$

2) LES of pair:

for $\bar{h}$ this is LES of $(X\times Y, A\times Y)$

$\underline{h}$ $H^*(Y)$ is direct sum of copies of R.

So $H^*(X,A)\otimes H^*(Y)$ is direct sum of copies of LES of $H^*(X,A)$

note: exact $\otimes$ free is exact)

3) Excision If $\bar{B} \subset \text{int} A \subset A \subset X$,

$j^* : \bar{h}(X,A) \longrightarrow \bar{h}(X-B, A-B)$ is an $\cong$ excision for $B\times Y \subset A\times Y \subset X\times Y$.

$\underline{i^*} : \underline{h}(X,A) \longrightarrow \underline{h}(X-B, A-B)$ " " excision for $B \subset A \subset C$

4) Collapsing a pair

If (X,A) is a good pair, $\pi : (X,A) \to (X/A, A/A)$

$\underline{\pi^*} : \underline{h}(X/A, A/A) \xrightarrow{\cong} \underline{h}(X,A)$, same for $\bar{h}$

Thm If X is an fcc, $\underline{\Phi} : \underline{h}(X) \xrightarrow{\cong} \bar{h}(X)$ is an iso

lemma: $\underline{\Phi}$ commutes with induced maps and δ map in LES of pair.

Proof only show commutes with induced maps.

to show commutes with δ, see ex.

suppose that $f : X_1 \to X_2$ $F : X_1 \times Y \longrightarrow X_2 \times Y$

$F = f \times 1_Y$

then $\bar{f}^*(\underline{\Phi}(a \otimes b))$

unsure this step $= F^*(a \times b)$

$\bar{f}^* : \bar{h} \to H^*(X\times Y, A\times Y)$

$= F^*(\pi_1^*(a) \cup \pi_2^*(b))$

$\underline{\Phi} : \underline{h} \to \bar{h}$

$= (F^* \pi_1^*(a)) \cup (F^*(\pi_2^*(b)))$

$= (\pi_1 \circ F)^*(a) \cup (\pi_2 \circ F)^*(b)$

unsure these steps $= \pi_1^* \circ f^*(a) \cup \pi_2^*(b) = f^*(a) \times b = \underline{\Phi}(\underline{f}^*(a \otimes b))$

Proof of the big thm: If X is an FCC, $\underline{\Phi}$ is an isomorphism

let $P(X,A)$ be the statement that $\underline{\Phi} : \underline{h}(X,A) \to \bar{h}(X,A)$ is an $\cong$.

Recall that $\underline{h}(X,A) = H^*(X,A) \otimes H^*(Y)$

$\bar{h}(X,A) = H^*(X\times Y, A\times Y)$

Proof steps

A) $P(\{\bullet\})$, $P(S^0)$ holds

B) If $X_1 \simeq X_2$, $P(X_1) \Leftrightarrow P(X_2)$

c) if two of $P(X)$, $P(A)$, $P(X,A)$ holds, the third holds

D) if (X,A) is a good pair, $P(X,A) \Leftrightarrow P(X/A)$

E) $P(S^n)$, $P(D^n, S^{n-1})$ holds

F) $P(X) \Rightarrow P(X \cup_f D^n)$

Proof Skipped,

Come back later.

Example. Compute $H^*(\Sigma_2)$

$\pi : \Sigma_2 \to \Sigma_2/A \cong T^2 \vee T^2$

recall, $H_2(\Sigma_2) \cong \mathbb{Z}$, $H_2(T^2 \vee T^2) = H_2(T^2) \oplus H_2(T^2) = \mathbb{Z} \oplus \mathbb{Z}$,

did we show this property?

$$\begin{tikzcd}
\pi_* : \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \\
1 \arrow[r, mapsto] & (1,1)
\end{tikzcd}$$

$\mathbb{Z}^4 = H_1(\Sigma_2) \xrightarrow{\cong} H_1(T^2 \vee T^2)$

$H_*(\Sigma_2)$, $H_*(T^2 \vee T^2)$ are free over $\mathbb{Z}$, UCT implies $H^*(\Sigma_2) = \mathrm{hom}(H_*(\Sigma_2), \mathbb{Z})$.

Same with $H^*(T^2 \vee T^2) = \mathrm{Hom}(H_*(T^2 \vee T^2), \mathbb{Z})$.

let π^* be dual to π_*.

$$\begin{tikzcd}
\pi^* : H^2(T^2 \vee T^2) \arrow[r] \arrow[d, equal] & H^2(\Sigma_2) \arrow[d, equal] \\
H^2(T^2) \oplus H^2(T^2) & \mathbb{Z} \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"] & \mathbb{Z} \\
\langle c_1, c_2 \rangle & \langle c \rangle
\end{tikzcd}$$

Scheme:
- get π^*
- think where π^* sends each generators to!

$$\begin{tikzcd}
\pi^* : H^1(T^2 \vee T^2) \arrow[r, "\cong"] \arrow[d, equal] & H^1(\Sigma_2) \\
H^1(T^2) \oplus H^1(T^2) & \\
\langle a_1', b_1' \rangle \oplus \langle a_2', b_2' \rangle \arrow[r] & \langle a_1, b_1, a_2, b_2 \rangle
\end{tikzcd}$$

$a_i = \pi^*(a_i')$, $b_i = \pi^*(b_i')$

so $a_i \cup b_j = \pi^*(a_i') \cup \pi^*(b_j') = \pi^*(a_i' \cup b_j') = \pi^*(\delta_{ij} c_i) = \delta_{ij} c$. i.e. $(a_1 \cup b_1)$, $(a_2 \cup b_2)$ give you c.

similarly, $a_i \cup a_j = 0$, $b_i \cup b_j = 0$

this gives us the ring structure of $H^*(T^2 \vee T^2)$.

Ex sheet 3. Same arguments show that

$H^1(\Sigma_g) = \langle a_i, b_i \rangle_{i=1}^{g}$ with $a_i \cup b_j = \delta_{ij} c$ $\quad \langle c \rangle = H^2(\Sigma_g) = \mathbb{Z}$ $\quad a_i \cup a_j = b_i \cup b_j = 0$.

defn E, B, π

① fibres
② local trividlivatins
ⓟ π_1 a) defined like an comm diagram.
ⓟ π_2 b) homeo per fibre

IV Vector bundles

defn. n-dime real vector bundle

An n-dime real vector bundle (E, B, π) is two spaces

s.t. 1) $\pi^{-1}(b) \sim \mathbb{R}^n \quad \forall b \in B$

2) there's an open cover $\{U_\alpha \mid \alpha \in A\}$ of B and maps

$f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ s.t.

E: total space
B: base
$\pi : E \to B$.

a)
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "f_\alpha"] \arrow[d, "\pi"] & U_\alpha \times \mathbb{R}^n \arrow[d, "\pi_1"] \\ U_\alpha \arrow[r, "Id_{U_\alpha}"] & U_\alpha \end{tikzcd}$$
commutes.

homeomorphisms.

b) $\pi_2 \circ f_\alpha : \pi^{-1}(b) \to \mathbb{R}^n$ is an $\cong$ of vector spaces for all $b \in U_\alpha$.

f_α are called local trividlisations

def complex vector bundles.

Same thing, replace $\mathbb{R}$ with $\mathbb{C}$.

def morphism

A morphism of vector bundles is a commuting square:

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f_B"] & B' \end{tikzcd}$$

s.t. $f_E|_{\pi^{-1}(b)} : \pi^{-1}(b) \to (\pi')^{-1}(f(b))$

$\mathbb{R}^n \to \mathbb{R}^m$ is linear

★ note: fibers can be of diff dimensions

def bundle isomorphism

bundle morphism $E_1 \to E_2$ with an inverse which is also a bundle hom $E_2 \to E_1$

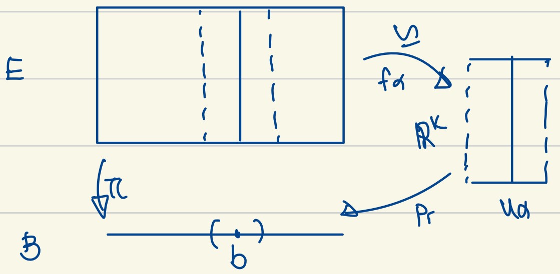
is a bundle $\cong$.

def. subbundle

E is a subbundle of E' if $\exists$ injective morphism

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "1_B"] & B' \end{tikzcd}$$

i.e. $\pi^{-1}(b)$ is a liner subspace of $(\pi')^{-1}(b)$.



Week 7 lec 1.

def. section

a section s of E is a cts map $s: B \to E$ with $\pi \circ s = \mathrm{Id}_B \Leftrightarrow s(b) \in \pi^{-1}(b)$

s is nonvanishing if $s(b) \neq \underbrace{0_b}_{\text{the 0 vector in } \pi^{-1}(b).}$ $\forall b$

def. continuous section

$s_0 : B \to E$ $b \mapsto 0_b$ is the 0 section.

to check if a section is cts, enough to check $f_\alpha \circ s$.

Ex. n-diml trivial bundle & trivial bundle.

$E = B \times \mathbb{R}^n$ $\pi : E \to B$ proj on B $f: E \to B \times \mathbb{R}^n$ is a local triv. $f = 1_{B \times \mathbb{R}^n}$

moreover, $\pi: E \to B$ is trivial if there's a bundle $\cong$ $f: E \to B \times \mathbb{R}^n$.

Prop. equivalent conditions of being a trivial bundle

E is trivial $\Leftrightarrow$ $\exists$ sections $s_1, \cdots s_n : B \to E$ s.t. $\{s_1(b), \cdots s_n(b)\}$ is a basis for $\pi^{-1}(b)$, $\forall b \in B$.

Proof $\Rightarrow$ we can find those sections explicitly.

$\Leftarrow$ the map $F: B \times \mathbb{R}^n \to E$

$(b, \vec{v}) \mapsto \sum_{i=1}^{n} v_i s_i(b)$ is a bundle $\cong$.

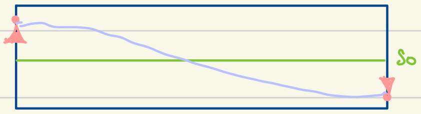
Ex. The Mobius Bundle

$$\begin{tikzcd} M = [0,1] \times \mathbb{R} / \sim \arrow[d] \\ S^1 = [0,1] / \sim \end{tikzcd}$$

$\sim$ is the smallest eq. rel with $(0,x) \sim (1,-x)$

A section $s: S^1 \to M$ gives $f: [0,1] \to \mathbb{R}$ with $f(0) = -f(1)$.

So $f(t) = 0$ for some $t \in [0,1]$. So it's not a nonvanishing section.



Ex. The tautological bundle

$$\begin{tikzcd} T_{\mathbb{RP}^n} = \{ ([\vec{z}], \vec{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid V \in \langle \vec{z} \rangle \} \arrow[d, "\pi"] \\ \mathbb{RP}^n \end{tikzcd}$$

$$\pi^{-1}([z]) = \underset{\mathbb{R}}{\overset{\wr\wr}{\langle \vec{z} \rangle}} \subset \mathbb{R}^{n+1}$$

have open cover $U_i = \{ [\vec{z}] \in \mathbb{RP}^n \mid z_i \neq 0 \}$.

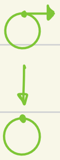
have maps $f_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}$

$([\vec{z}], \vec{v}) \longrightarrow ([z], v_i)$ are local trivialisations

$\mathbb{RP}^1 = S^1$ and $T_{\mathbb{RP}^1} = M$ is nontrivial.

Similarly, $T_{\mathbb{CP}^1}$ a 1-diml cx VB over $\mathbb{CP}^n$.

Ex Tangent sphere bundles



$$\begin{tikzcd} TS^n = \{ (\vec{x}, \vec{v}) \in S^n \times \mathbb{R}^{n+1} \mid \vec{v} \cdot \vec{x} = 0 \} \arrow[d, "\pi"] \\ \vec{x} \in S^n \end{tikzcd}$$

tangent to S^n

local trivialisations

$\pi^{-1}(x) = x^{\perp} \simeq \mathbb{R}^n$

$U_i = \{ x \in S^n \mid x_i \neq 0 \}$

$f_i^{-1} = \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$

$(\vec{x}, \vec{v}) \longrightarrow (\vec{x}, \pi_{\hat{i}} \vec{v})$ → drop the ith coordinate

$\vec{v}$ is a $n+1$ diml vector, $\pi_{\hat{i}} \vec{v}$ has one less diml.

↳ TS^1 has a non-vanishing section $s((x,y)) = ((x,y),(-y,x))$ ⇒ TS^1 is trivial.

↳ TS^{2n} has no non-vanishing section, so it's nontrivial. ? Proof?

def Pullbacks of vector bundle

$$\begin{tikzcd} & E \arrow[d] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

let $\pi : E \to B$ be a n-diml r.VB., $g. B' \to B$ continuous.

then the pullback of E by g is

$g^*(E) = \{ (b', b, v) \in B' \times B \times E \mid g(b') = b = \pi(v) \}$

$\pi_g : g^*(E) \to B'$

$(b', b, v) \to b'$

$\pi_g^{-1}(b') = \{ (b', g(b'), v) \mid \pi(v) = g(b') \} = \pi^{-1}(g(b'))$ is a vec space.

If $f_\alpha: \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ is a local triv for E.

let $V_\alpha = g^{-1}(U_\alpha)$

then $f'_\alpha : \pi_g^{-1}(V_\alpha) \longrightarrow V_\alpha \times \mathbb{R}^n$ is a local trivialisation for $g^*(E)$.

$(b, b', v) \mapsto (b', \pi_2(f_\alpha(v)))$

lemma $(g \circ f)^*(E) = f^*(g^*(E))$

def restriction to a smaller base

If $A \subset B$, $i: A \hookrightarrow B$, is the inclusion, then, $E|_A := i^*(E)$ is the restriction of E to A.

lem. non-vanishing section could be "pulled back"

$s: B \to E$ a non-vanishing section. Then

$g^*s : B' \longrightarrow g^*(E)$

$b' \mapsto (b', f(b), s(f(b)))$ is a non-vanishing section of $g^*(E)$.

Example : $\mathbb{RP}^n$ restrict to $\mathbb{RP}^1$

$T\mathbb{RP}^n|_{\mathbb{RP}^1} \simeq T\mathbb{RP}^1$ has no non-vanishing section (i.e. if it did, $\mathbb{RP}^1$ would have as well).

$\Rightarrow T\mathbb{RP}^n$ has no non-v section

$\Rightarrow T\mathbb{RP}^n$ is nontrivial.

defn. Products of two vector bundles

$\pi: E \to B$, $\pi': E' \to B'$ of dim n, n' respectively.

then their product is a vector bundle of dim $n+n'$,

$(\pi \times \pi')^{-1}(b, b') = \pi^{-1}(b) \times \pi'^{-1}(b') \subseteq E \times E'$

their local trivialisations are as follows:

If $\left\{ \begin{array}{l} f_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n \\ f'_\beta : (\pi')^{-1}(U_\beta) \longrightarrow U_\beta \times \mathbb{R}^{n'} \end{array} \right\}$ are local trivialisations,

then, $f_\alpha \times f'_\beta : (\pi \times \pi')^{-1}(U_\alpha \times U_\beta) \longrightarrow U_\alpha \times \mathbb{R}^n \times U_\beta \times \mathbb{R}^{n'} \cong U_\alpha \times U_\beta \times \mathbb{R}^{n+n'}$

is a local trivialisation of $E \times E'$.

def Whitney sum

If $B = B'$, $E \oplus E' = \Delta^*(E \times E')$ where $\Delta: B \to B \times B$ is the Whitney sum of E and E'.

$$\begin{tikzcd} \Delta^*(E \times E') \arrow[d] \\ B \end{tikzcd}$$

$b \mapsto (b, b)$

def supp

$\varphi : B \to \mathbb{R}$, $\quad \mathrm{Supp}(\varphi) = \overline{\{b \in B \mid \varphi(b) \neq 0\}}$

def. Partition of unity. subordinate to a cover

let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of B, a PoU subordinate to $\mathcal{U}$ is a fam of fns

$\varphi_i : B \to \mathbb{R} \quad (i \geq 0)$

s.t. 1) $0 \leq \varphi_i(b) \leq 1$

2) $\{i \mid \varphi_i(b) \neq 0\}$ is finite $\forall b \in B$

3) $\mathrm{Supp}\, \varphi_i \subset U_{\alpha_i}$ for some $\alpha_i \in A$.

important bit

4) $\sum_{i \geq 0} \varphi_i(b) = 1 \quad \forall b \in B$

def. admits PoU

B admits a PoU if B admit a POU subordinate to all open covers $\mathcal{U}$.

note: compact or metrizable ⇒ B admit PoU

para 〃 〃 〃 ⇒ 〃 〃

Thm. $E|_{B\times 0} \cong E|_{B\times 1}$

Suppose that B admits PoU. $\pi : E \to B \times I$ is a rVB. Then $E|_{B\times 0} \cong E|_{B\times 1}$.

Week 7 lec 3 (lec 20)

lemma 1. If $E|_{B\times[0,1/2]}$ and $E|_{B\times[1/2,1]}$ are trivial then E is trivial.

Proof proof ?

lemma 2. for each $b \in B$, b has an open nbhd U_b s.t. $E|_{U_b \times I}$ is trivial.

Proof

E locally trivial ⇒ for each $t \in I$, we can find

$$\left.\begin{array}{l} U_t \text{ an open nbhd of } b \text{ in } B \\ I_t \text{ an open nbhd of } t \text{ in } I. \end{array}\right\} \text{ s.t. } E|_{U_t \times I_t} \text{ is trivial.}$$

$\{I_t \mid t \in I\}$ is an open cover of I. let $\{I_{t_0}, \cdots, I_{t_n}\}$ be a finite subcover.

then $\exists\ 0 = s_0 < s_1 < \cdots < s_n = 1$ s.t. $[s_i, s_{i+1}] \subset I_{t_k}$ for some k. (Lesbesgue covering lemma) so, $E|_{U_{t_k}\times[s_i,\,s_{i+1}]}$ is trivial.

let $U_b = \bigcap_{k=0}^{n} U_{t_k}$ be open nbhd of b. (it's a finite intersection) and $E|_{U_b\times[s_i,s_{i+1}]}$ is trivial for all i.

By lemma 1 and induction, $E|_{U_b\times[0,s_i]}$ is trivial. then use lemma 1 and induction, we get $E|_{U_b\times[0,1]}$ is trivial.

Proof of thm ($E|_{B\times\{0\}} \cong E|_{B\times\{1\}}$)

PoU indexed by $\mathbb{N}$?

let U_b be as in lemma 2. Pick a PoU $\boxed{\{\varphi_i\}_{i=1}^{\infty}}$ subordinate to $\{U_b \mid b\in B\}$.

Suppose that Supp $\varphi_i \subset U_{b_i}$. What is U_{b_i}?

let $g_k : B \to B\times I$

$$b \mapsto (b, \underbrace{\textstyle\sum_{i=1}^{k}\varphi_i(b)}_{\substack{\shortparallel \\ \psi_i(b)}})$$

and define $E_k = g_k^*(E) = \{(b, \underbrace{(b,\psi_k(b))}_{g_k(E)}, v) \mid \pi(v) = (b,\psi_k(b))\}$

let $f_i : (U_b\times I) \longrightarrow U_b \times I \times \mathbb{R}^n$ be a trivialisation.

Define $\beta_k : E_{k-1} \longrightarrow E_k$ by

$$\beta_k((b, g_k(b), v)) = \begin{cases} (b, g_k(b), v) & \text{for } b\notin U_{b_k} \\ (b, f_k^{-1}(b, g_k(b), v)) & \text{for } b\in U_{b_k}. \end{cases}$$

where $f_k(v) = (b, g_{k-1}(b), v')$

then $\cdots \circ \beta_3 \circ \beta_2 \circ \beta_1$ is the desired isomorphism $E|_{B\times 0} \longrightarrow E|_{B\times 1}$ (for each point, it stabilizes.)

Don't understand the proof !

Proof scheme

$g_k : B \to B\times I$

$b \mapsto (b, \sum_{i=1}^{k}\varphi_i(b))$

$\beta_k : E_k \to E_{k+1}$

compose $\cdots \circ \beta_3 \circ \beta_2 \circ \beta_1$ $E|_{B\times 0} \to E|_{B\times 1}$.

Cor Suppose $\pi: E \longrightarrow B$ is a VB. $g_0, g_1: B' \longrightarrow B$. $g_0 \sim g_1$ via $h: B' \times I \longrightarrow B$, and B' admits a PoU.

then $g_0^*(E) = h^*(E)|_{B' \times 0} \cong h^*(E)|_{B' \times 1} = g_1^*(E)$.

Cor: If B is contractible, and admits a PoU, then every VB $\pi: E \to B$ is trivial.

Proof: $1_B \sim C_{B,p}$, $E = (1_B)^* E \cong (C_{B,p})^* E = B \times \pi^{-1}(p)$ is trivial.
$p \in B$

↑ Where copy of B?

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ p \arrow[r, "i_{B,p}"'] & B \end{tikzcd}$$

$(i_{B,p})^* E = \{ (b', b, v) \in \{p\} \times B \times E \mid g(b') = \pi(v) = b \}$

$= (p, p, \pi^{-1}(p))$? Why $B \times \pi^{-1}(p)$ not $\pi^{-1}(p)$

Riemannian Metrics

def Riemannian Metrics

Suppose $\pi: E \longrightarrow B$ is a rVB (resp. cx VB)

A Riemannian (resp. Hermitian) metric on E is a continuous map

$g: E \oplus E \longrightarrow \mathbb{R}$ (resp. $E \oplus E \to \mathbb{C}$)

s.t. $g|_{\pi^{-1}_{E \oplus E}}$ is an inner product (resp. hermitian inner product)

$\pi^{-1}_{E \oplus E}(b) = \pi^{-1}(b) \times \pi^{-1}(b)$ on $\pi^{-1}(b) \times \pi^{-1}(b)$.

Ex $T\mathbb{RP}^n = \{ ([z], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \langle z \rangle \}$

has a natural Riemannian metric given by $g(([z], v_1), ([z], v_2)) = \langle v_1, v_2 \rangle_{\mathbb{R}^{n+1}}$

Similarly $T\mathbb{CP}^n$ has a natural Hermitian metric.

def unit disk, unit sphere bundle

Suppose E is a v.b with Riemannian metric g.

The unit disk, the unit sphere bundle are given by

$S_g(E) = \{ v \in E \mid \langle v, v \rangle = 1 \}$ $\pi: S_g(E) \to B$ $\pi^{-1}(b) \cong S^{n-1}$

$D_g(E) = \{ v \in E \mid \langle v, v \rangle \leq 1 \}$ $\pi: D_g(E) \to B$ $\pi^{-1}(b) \cong D^n$

they are top spaces but not r VBs.

Prop. If g, g' are 2 R-metrics on E, then.

$$\begin{tikzcd} S_g(E) \arrow[rr, phantom, "\cong"] \arrow[dr, "\pi"] & & S_{g'}(E) \arrow[dl, "\pi"'] \\ & B & \end{tikzcd}$$

similarly

$$\begin{tikzcd} D_g(E) \arrow[rr, phantom, "\cong"] \arrow[dr, "\pi"'] & & D_{g'}(E) \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

So we can drop g from our notation. Write $S(E)$, $D(E)$

$\uparrow$ i.e. do not depend on the inner prod.

Proof. Excercise. ???

Example

$$S(\underset{\text{is } S^n}{T_{\mathbb{R}P^n}}) = \{([z], v) \mid \underset{\text{is } v \in S^n}{\|v\|_{\mathbb{R}^{n+1}} = 1}, v \in \langle z \rangle\}$$

$$\begin{tikzcd} \pi : S(T_{\mathbb{R}P^n}) \arrow[r] \arrow[d, phantom, "||"] & \mathbb{R}P^n \\ S^n \arrow[ur, "\text{natural projection}"'] & \end{tikzcd}$$

$$\begin{tikzcd} S(T_{\mathbb{C}P^n}) \arrow[r, phantom, "= S^{2n-1}"] \arrow[d, "\pi"] & {} \arrow[dl, "\text{Hopf map}"] \\ \mathbb{C}P^n & \end{tikzcd}$$

Ex. If $\pi : E \to B$ is trivial, then E has an R-metric ($\swarrow$ Riemannian) gives by $g(v_1, v_2) = \langle \pi_2(f(v_1)), \pi_2(f(v_2)) \rangle$

$$\begin{tikzcd} E \arrow[d, "f"] \\ B \times \mathbb{R}^n \end{tikzcd}$$

$\Rightarrow S(B \times \mathbb{R}^n) = B \times S^{n-1}$ (if it were trivial.)

$\Rightarrow T\mathbb{R}P^n$, $T\mathbb{C}P^n$ are nontrivial since $\mathbb{R}P^n \times S^0 \not\cong S^n$ (homology group)

$\mathbb{C}P^n \times S^1 \not\cong S^{2n-1}$

Prop. Base has PoU then it has R-metic

If B admits a PoU, $\pi : E \to B$ is a rVB, then E has an R-metric.

Proof:

B has an open cover $\{U_\alpha \mid \alpha \in A\}$ s.t. $E|U_\alpha$ is trivial, so there's an R-metric g_α on $E|U_\alpha$. Choose PoU subordinate to U_α. take $g = \sum_i \varphi_i g_{\alpha_i}$ where $\operatorname{supp} \varphi_i \subset U_{\alpha_i}$.

The Thom Isomorphism

$\pi : E \to B$ is n-diml VB.

If $b \in B$, let $\underset{\text{is } \mathbb{R}^n}{E_b = \pi^{-1}(b)}$ be the fibre of E over b.

$i_b : E_b \hookrightarrow E$ inclusion

$s_0 : B \longrightarrow E$ 0-section

Define $E^\# = E \setminus \text{im } s_0$

$E^\#_b = E_b \setminus 0$

Then, $H^*(E_b, E_b^\#) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - 0) = \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{o.w.} \end{cases}$ is free.

By UCT, $H^*(E_b, E_b^\#; R) = \begin{cases} R & *=n \\ 0 & \text{o.w.} \end{cases}$

$i_b : (E_b, E_b^\#) \to (E, E^\#)$

$i_b^* : H^*(E, E^\#; R) \to H^*(E_b, E_b^\#; R)$.

defn. R-Thom class

$u \in H^n(E, E^\#; R)$ is an R-Thom class for E if $i_b^*(u)$ generates $H^*(E_b, E_b^\#; R)$ for all $b \in B$.

$\left(H^n(E_b, E_b^\#; R) = H^n(\mathbb{R}^n, \mathbb{R}^n - 0; R) \cong R \quad \forall b \in B \right)$

Week 8 lecture 1

(recall) * assume R-coeff.

Ex. E is trivial, $E = B \times \mathbb{R}^n$

$$H^*(E, E^\#) = H^*(B \times \mathbb{R}^n; B \times (\mathbb{R}^n \setminus \{0\})) \cong H^*(B) \otimes \overbrace{\underbrace{H^*(\mathbb{R}^n, \mathbb{R}^n - 0)}_{\text{using fact it's free over } \mathbb{R}.}}^{\text{generated by 1 thing. Hence rest is iso.}}$$

? (ie $H^{k-n}(B) \xrightarrow{\cong} H^k(E, E^\#)$

$a \longmapsto a \times u = \pi_1^*(a) \cup \pi_2^*(u)$, $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong R$.

u generates $H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$

Also note that $H^0(B) = \prod_{B_i \in \pi_0(B)} H^0(B_i)$ so we can specify $\vec{r} \in H^0(B)$ by a tuple $r = (r_1, \cdots, r_k)$ for $r_i \in H^0(B_i)$.

↑ i-th path component.

In particular,

$H^0(B) \cong H^n(E, E^\#)$

$\vec{r} \mapsto r \times u$ expand × then include i_b^* note $i_b^*(u) = H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$

If $b \in B_i$, $i_b^*(\vec{r} \times u) = r_i \cup H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ → want this to be $H^*(B)$

so $\vec{r} \times u$ is a Thom class $\iff$ r_i generates $H^0(B_i)$ for all i.

If $R = \mathbb{Z}/2$, there is a unique Thom class

If $R = \mathbb{Z}$ there are $2^{|\pi_0(B)|}$ Thom classes (choose $r_i = \pm 1$).

Pullbacks

If $f: B' \to B$ there's a morphism of vector bundles

$(b', b, v) \mapsto v$

$$\begin{tikzcd} f^*(E) \arrow[r, "F"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

note: $F(\text{Im}\, s_0') = \text{Im}\, s_0$ so F is a map of pairs $(f^*E, f^*E^\#) \longrightarrow (E, E^\#)$

lemma 1 If u is a R-TC for E then F^*u is an R-TC for f^*E.

Proof there's a commuting square

$$\begin{tikzcd} f^*(E) \arrow[r, "F"] & E \\ (f^*E)_{b'} \arrow[u, "i_{b'}"] \arrow[r, "F|_{f^*(E)_{b'}}", "\cong"'] \arrow[ur, phantom, "=j"] & E_{f(b')} \arrow[u, "i_{f(b')}"'] \\ \mathbb{R}^n \arrow[r, mapsto] & \mathbb{R}^n \end{tikzcd}$$

so $i_{b'}^*(F^*(u)) = j^*(i_{f(b')}^*(u))$, j^* is an $\cong$ and $i_{f(b')}^*(u)$ generates $H^*(E_{f(b')}, E_{f(b')}^\#)$ so $i_{b'}^*(F^*(u))$ generates

$\Rightarrow F^*(u)$ is a TC.

i.e. TC behaves naturally under pullbacks.

lemma 2

Suppose that $B = B_1 \cup B_2$, $u \in H^n(E, E^\#)$

$i_k: B_k \to B$ be inclusion. If $i_1^*(u)$ are TC for $E|_{B_1}$

$i_2^*(u)$ " " $E|_{B_2}$

then u is a TC for E.

Proof Exercise ???

Thm (important) Thom $\cong$

If $\pi: E \to B$ is an n-dim'l r-VB then

1) E has a unique $\mathbb{Z}/2$ Thom class

2) If E has an R-Thom class u, the map

$$\Phi: H^*(B;R) \longrightarrow H^{*+n}(E, E^\#; R) \text{ is an } \cong$$

$$a \longmapsto \pi^*(a) \cup u$$

Proof: (skipped. Come back later).

Come back later !

* * *

Gysin Sequence :

Suppose $\pi : E \to B$ has an R Thom class u.

Note: $E^{\#} = E \setminus \operatorname{im} s_0 \sim S(E)$

$$v \mapsto v / \sqrt{g(v,v)}$$

LES of $(E, E^{\#})$ is $j : (E, \phi) \to (E, E^{\#})$

$$\begin{tikzcd}
H^*(E, E^{\#}) \arrow[r, "j^*"] \arrow[dr, bend left, "\text{Euler class}"] & H^*(E) \arrow[r] \arrow[d, "\cong"] \arrow[d, bend right, "s_0^* \text{ is } \cong"'] & H^*(E^{\#}) \arrow[r] \arrow[d, phantom, "\cong"] & H^{*+1}(E, E^{\#}) \\
H^{*-n}(B) \arrow[r, "\alpha"] \arrow[u, "\Phi", "\cong"'] & H^*(B) \arrow[r] \arrow[u, bend right, "\pi^* \text{ is } \cong"'] & H^*(S(E)) \arrow[r] & H^{*-n+1}(B) \arrow[u, "\Phi", "\cong"']
\end{tikzcd}$$

$\pi : E \to B$ and $s_0 : B \to E$ give homotopy equivalence. They're homotopy inverses.

then, $\alpha(a) = s_0^* j^* (\Phi(a))$

$= s_0^* j^* (\pi^* a \cup u)$

$= s_0^* (\pi^* a \cup j^* u)$ ← Why j only show up 2nd part cup prod ?

$= (s_0^* \pi^* a) \cup s_0^* j^* (u) = a \cup s_0^* j^* (u)$

Def Euler class

If $\pi : E \to B$ is an R- oriented n-diml rVB with TC u.

then its Euler class is $e(E) = s_0^* j^* (u) \in H^n(B)$

Thm. (Gysin Sequence)

There's an LES

$$\longrightarrow H^{*-n}(B) \xrightarrow{\alpha} H^*(B) \xrightarrow{\pi^*} H^*(S(E)) \longrightarrow H^{*-n+1}(B)$$

where $\alpha(a) = a \cup e(E)$

Proof : Basically comes from the LES of $(E, E^{\#})$ with thom iso.

Week 8 lec 2 (lec 22) $\underbrace{\text{underlying}}$ ring for coho is R

Recall: $\pi : E \to B$ is an R- oriented n-diml r-VB, with TC $u \in H^*(E, E^{\#})$

then its Euler cl is $e(E) = s_0^* j^* (u)$ where $\begin{cases} s_0 : B \to E \text{ is } 0 \text{ section} \\ j : (E, \phi) \to (E, E^{\#}) \text{ inclusion of pairs} \end{cases}$

$e(E)$ goes from $H^*(E, E^{\#})$ to $H^*(B)$.

Thom iso Φ "bumps up" coho of B to coho of $(E, E^{\#})$

$e(E)$ is the thom class going the other way.

Prop: (Properties of e)

Suppose E as above, then

1) $f: B' \to B$ then $f^*(E)$ is oriented and $e(f^*(E)) = f^*(e(E))$

2) If E is trivial and $n>0$, then $e(E) = 0$

3) $e(E_1 \oplus E_2) = e(E_1) \oplus e(E_2)$

4) If E has a nonvanishing section, then $e(E) = 0$.

relationship btwn trivial & nonvanishing section.

Proof:

1) There is a commuting diagram

$$\begin{tikzcd}
(B,\emptyset) \arrow[r, "s_0"] & (E,\emptyset) \arrow[r, "j"] & (E, E^{\#}) \\
(B',\emptyset) \arrow[u, "f"] \arrow[r, "s_0'"] \arrow[urr] & (f^*E, \emptyset) \arrow[u, "f_E = F"] \arrow[r, "j'"] & (f^*E, (f^*E)^{\#}) \arrow[u, "f_E = f"]
\end{tikzcd}$$

$= R$-Thom class

By lemma 1, $f_E^*(u)$ is an orientation on $f^*(E)$, so

$$e(f^*(E)) = s_0'^* j'^* f_E^*(u) = j^* s_0^* f^*(u) = f^*(e(u))$$

2) True if $B = \{0\}$ since $H^n(\{0\}) = 0$ (i.e. $u \in H^n(E, E^{\#})$ trivial so its $s_0^* j^*(u)$ must be).

in general, E is trivial $\Leftrightarrow E = f^*(E_0)$ wher $f: B \to \{0\}$, $E_0 = \mathbb{R}^n$, $\pi: \underset{\mathbb{R}^n}{\underset{\shortparallel}{E_0}} \to 0$

$$\begin{tikzcd}
 & E_0 = \mathbb{R}^n \arrow[dl] \arrow[d, "\pi"] \\
B \arrow[r, "f"] & \{0\}
\end{tikzcd}$$

note: $f^*(E_0) = \{ b, \cdot, e \} = B \times \mathbb{R}^n$

$\boxed{\underbrace{s_0^*}_{} j^*(u) \text{ in } E_0}$

codomain is $H^*(B)$ = trivial.

$$e(E) = e(f^*(E_0)) = f^*(e(E_0)) = f^*(0) = 0$$

3) Ex Sheet 4

4) If s is an nonvanishing section, $E = \overset{\text{trivial bundle.}}{\langle s \rangle} \oplus \langle s \rangle^{\perp}$ (subspace generated by s)

$$\Rightarrow e(E) = e(\langle s \rangle \oplus \langle s \rangle^{\perp}) = \underbrace{e(\langle s \rangle) \cup e(\langle s \rangle^{\perp})}_{\oplus \text{ of coho from diff grading thus have to use cup ?}} = 0 \cup e(\langle s \rangle^{\perp}) = 0$$

Recall Gysin Sequene:

$$H^{*-n}(B) \xrightarrow{\alpha} H^*(B) \xrightarrow{\pi_{S(E)}^*} H^*(S(E)) \longrightarrow H^{*-n+1}(B)$$

where $S(E) \xrightarrow{\pi_{SE}} B$

where $\alpha(a) = a \cup e(E)$

(note: UCT help us to figure out $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$ as group. free over $\mathbb{Z}/2$ now need Gysin sequence to figure out ring structure).

Thm: Solving cohomology $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$

$H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/x^{n+1}$ where $x = e(T_{\mathbb{RP}^n}) \in H^1(\mathbb{RP}^n;\mathbb{Z}/2)$

(every VB in $\mathbb{Z}/2$ is orientable / admits a thom class)

Proof We assume that we're using $\mathbb{Z}/2$ coefficients everywhere.

$S(T_{\mathbb{RP}^n}) = S^n$ sphere bundle. (did we prove this?)

Recall (H_* + UCT):

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \le * \le n. \\ 0 & \text{o.w.} \end{cases}$$

Gysin Sequene: go up in grading every 3 rings

$$H^{k-1}(\mathbb{RP}^n) \xrightarrow{\alpha} H^k(\mathbb{RP}^n) \longrightarrow H^k(S^n) \longrightarrow H^k(\mathbb{RP}^n) \longrightarrow$$

any nonzero of appropriate grading

Claim: $\alpha = \bullet \cup x$ is an $\cong$ for $1 \le k \le n$. $(*)$

$k=1$: (write it longer so includes $k=0$ as well)

$$\longrightarrow \underset{=0}{H^{-1}(\mathbb{RP}^n)} \xrightarrow{\alpha} \underset{=\mathbb{Z}/2}{H^0(\mathbb{RP}^n)} \xrightarrow{\cong \text{ since inj}} \underset{=\mathbb{Z}/2}{H^0(S^n)} \xrightarrow{0} H^0(\mathbb{RP}^n) \xrightarrow{\alpha} H^1(\mathbb{RP}^n) \longrightarrow \underset{=0}{H^1(S^n)}$$

since previs inj

hence α is an $\cong$.

$1 < k < n$:

$$\underset{=0}{H^{k-1}(S^n)} \longrightarrow H^{k-1}(\mathbb{RP}^n) \xrightarrow[\cong]{\alpha} H^k(\mathbb{RP}^n) \longrightarrow \underset{=0}{H^k(S^n)}$$

$k=n$

$$\underset{=0}{H^{n-1}(S^n)} \longrightarrow H^{n-1}(\mathbb{RP}^n) \xrightarrow[\cong]{} H^n(\mathbb{RP}^n) \xrightarrow{0 \text{ b/c RHS}} \underset{=\mathbb{Z}/2}{H^n(S^n)} \xrightarrow{\cong \text{ surj so iso}} \underset{=\mathbb{Z}/2}{H^n(\mathbb{RP}^n)} \longrightarrow \underset{=0}{H^{n+1}(\mathbb{RP}^n)}$$

By induetion, $(*) \longrightarrow \langle x^k \rangle$ generates $H^k(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \le k \le n$

$x^{n+1} \in H^{n+1}(\mathbb{RP}^n) = 0$

Scheme: $\boxed{\alpha: H^{k-1}(\mathbb{RP}^n) \longrightarrow H^k(\mathbb{RP}^n) \text{ is } \cong \text{ via Gysin. } 1 \le k \le n.}$

■

(ES 3)

Similarly, $T_{\mathbb{C}P^n}$ is a VB $\Rightarrow$ underlying r-VB is $\mathbb{Z}$-orientable. So $S(T_{\mathbb{C}P^n}) = S^{2n-1}$

Same argument shows

thm: $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$, $x = e(T_{\mathbb{C}P^n}) \in H^2(\mathbb{C}P^n; \mathbb{Z})$.

Verify.

Cor. $\pi_3(S^2) \neq 0$

$f: S^3 \to$ space.

Proof if $\pi_3(S^2) = 0$ then $\mathbb{C}P^2 \sim S^2 \vee S^4$

(since $\mathbb{C}P^2 = S^2 \cup_h D^4$ $h: S^3 \to S^2$ Hopf map. The attaching map is null-homotopic (i.e if $\pi_3(S^2) = 0$) so $\mathbb{C}P^2 = S^2 \cup_h D^4 \simeq S^2 \vee S^4$)

But if $x \in H^2(S^2 \vee S^4)$, $x \cup x = 0$ ■

(via coho of wedge product) (as before)

Comments on Orientability

1) Every E is $\mathbb{Z}/2$ orientable.

2) for $p \neq 2$, E is $\mathbb{Z}/p$ orientable $\Leftrightarrow$ E is $\mathbb{Z}$-orientable.

(If so, just say E is orientable)

3) $T\mathbb{R}P^1 = M$ is not orientable.

Since $H^*(M, M^{\#}) \cong H^*(D(M), S(M)) \cong H^*(\bar{M}, \partial\bar{M})$ M = closed mobius band.

$H^2(\bar{M}, \partial\bar{M}) = \mathbb{Z}/2 \not\cong H^1(S^1)$ so Thom $\cong$ with $\mathbounded{Z}$-coefficient is false.

↑ boundary of M include into M twice?

4) There's a homomorphism (ES4) $\varphi: \pi_1(B) \to \mathbb{Z}/2$ $\gamma: S^1 \to B$

$\varphi([\gamma]) = 0 \Leftrightarrow \gamma^*(E)$ is orientable.

If $\pi_1(B) = \{1\}$, any $\pi: E \to B$ is orientable.

V Manifolds

5.1) Definitions + Fundamental class

Def n-manifold

An n-manifold is a 2nd countable Hausdorff space M with an open cover $\{U_\alpha \mid \alpha \in A\}$ and homeomorphisms $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$.

The transition functions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}: \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are homeomorphisms:

M is <u>Smooth</u> if φ_α's can be chosen s.t. $\varphi_{\alpha\beta}$ s are diffeomorphisms.

note: Any smooth manifold has a tangent bundle $\pi: TM \to M$, an n-diml vector bundle.

Fundamental class

def. Notations on fundamental class

If $\underset{\text{compact}}{A \subset M}$, write $(M|A) = (M, M-A)$

If $B \subset A$, $\iota: (M|A) \longrightarrow (M|B)$ is inclusion of pairs.

$(M, M-A) \longrightarrow (M, M-B)$

If $w \in H_*(M|A)$, $w|_B = \iota_*(w)$

Prop. Compute $H_*(M|x\,; R)$

If $x \in M$, $x \in U_\alpha \cong \mathbb{R}^n$ for some $\alpha \in A$.

then, by excision, $H_*(M|x) \cong H_*(U_\alpha|x) \overset{\varphi_*}{\cong} H_*(\mathbb{R}^n|\varphi(x)) = H_*(\mathbb{R}^n, \mathbb{R}^n - \varphi(x)) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{o.w.} \end{cases}$

$\Rightarrow H_*(M|x\,; R) \cong \begin{cases} R & * = n \\ 0 & \text{o.w.} \end{cases}$

def R- fundamental class

An R- fundamental class for $(M|A)$ is $w \in H_n(M|A\,; R)$ s.t. $w|_x$ generate $H_n(M|x)$ for all $x \in A$.

(it's an analogue of Thom class).

Thm. unique $\mathbb{Z}/2$ fundamental class

If $A \subset M$ is compact, $(M|A)$ has a unique $\mathbb{Z}/2$ fundamental class.

(note: Most interested in case when M is compact (M is closed))

A fundamental class for $(M|M) = (M, \emptyset)$ will be written as $[M] \in H_n(M)$.

Proof: Similar to Thom $\cong$, See Moodle.

def : orientable

M is orientable if it has an $\mathbb{Z}$- fundamental class.

week 8 lec 3 (lec 23)

defn. submanifold

$N \subset M$ is a k-diml smooth submani of an n-mani M if for every $x \in N$,
there is a smooth chart $\varphi_x : U_x \to \mathbb{R}^n$ s.t.

$$\varphi_x(U_x \cap N) \to \mathbb{R}^k \times 0 \subset \mathbb{R}^n$$

If $N \subset M$ is a smooth submani, $TN \subset TM|_N$ (subbundle)

def Normal bundle

$N \subset M$ is a smooth submani. Then $\nu_{M/N} = TN^{\perp} \subset TM|_N$ is the normal bundle of N in M.

$\Rightarrow$ $TM|_N = \nu_{M/N} \oplus TN$

Thm. Tubular nbhd thm

If $N \subset M$ is a closed smooth submani, then there's an open $V \subset M$, $N \subset V$ with
$(V, N) \cong (\nu_{M/N}, s_0 \nu_{M/N})$

lemma. Suppose that $E = E_1 \oplus E_2$ is orientable. Then E_1 is orientable $\Leftrightarrow$ E_2 is orientable.

Proof. Example Sheet.

$\exists\, \mathbb{Z}$- fund. cl $\qquad\qquad$ $\exists\, \mathbb{Z}$- Thom Cl.

Prop. M is orientable $\Leftrightarrow$ TM is orientable

Proof (sketch)

If $\gamma : S^1 \hookrightarrow M$, let $\nu(\gamma)$ be a Tubular nbhd.

M orientable $\Leftrightarrow$ $\nu(\gamma)$ is orientable for all γ

$\Leftrightarrow$ $\nu M/\gamma$ " "

$\Leftrightarrow$ $TM|_\gamma$ " "

$\Leftrightarrow$ TM " "

Cor. M orientable $\Leftrightarrow$ Normal bundle orientable

If M is orientable, $N \hookrightarrow M$ is a closed smooth submani.

M orientable $\Leftrightarrow$ $\nu_{M/N}$ orientable.

5.2. Poincaré Duality

Remark: Change coefficients and taking duals

From now, coefficients are in a field $\mathbb{F}$. i.e. $H^k(X) = H^k(X ; \mathbb{F})$

$\Rightarrow H^k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{F})$ By UCT, as $\mathbb{F}$ is a field.

$\mathrm{Hom}(H^k(X), \mathbb{F}) \xrightarrow[\varphi]{\cong} H_k(X)$ (double dual)

where $\boxed{\langle a, \varphi(\alpha) \rangle = \alpha(a)}$. $\alpha \in \mathrm{Hom}(H^k(X), \mathbb{F})$, $\varphi(\alpha) \in H_k(X)$, $a \in H^k(X)$, $\alpha(a) \in \mathbb{F}$.

If $a \in H^k(X)$, $a \cup \bullet : H^\ell(X) \to H^{\ell+k}(X)$ Same as $\alpha(a)$?

def: Cap product

$\bullet \cap a : H_{\ell+k}(X) \to H_\ell(X)$ is the dual of $a \cup \bullet$. note: $a \in H^k(X)$.

$\boxed{\langle b, x \cap a \rangle = \langle a \cup b, x \rangle}$ $a \in H^k(X)$, $b \in H^\ell(X)$, $x = H_{\ell+k}(X)$, $x \cap a = H_\ell(X)$

↳ $a \cup b \in H^{k+\ell}(X)$ $\varphi^{-1}(x) = \mathrm{Hom}(H^{\ell+k}(X), \mathbb{F})$ RHS $\in \mathbb{F}$.

↳ $\varphi^{-1}(x \cap a) = \mathrm{Hom}(H^\ell(X) ; \mathbb{F})$ so LHS $\in \mathbb{F}$.

sec. Intersection Pairing

Suppose that M is an $\mathbb{F}$-oriented n-manifold with Fund $[M] \in H_n(M)$.

def. Intersection pairing

the intersection pairing $(\cdot , \cdot) : H^k(M) \times H^{n-k}(M) \to \mathbb{F}$ is the bilinear pairing given by

$$\boxed{(a, b) = \langle \underset{\in H^n}{a \cup b}, \underset{\in H_n}{[M]} \rangle}$$

Satisfying $(b, a) = (-1)^{|a||b|} (a, b)$

$= (-1)^{k(n-k)} (a, b)$

If $\boxed{a \in H^k(M), (a, \cdot) \in \mathrm{Hom}(H^{n-k}(M), \mathbb{F})}$

def. algebraic poincare dual (Big PD)

the algebraic poincare dual of a is $\boxed{PD(a) = \varphi((a, \bullet)) = \overset{H_n(x)}{[M]} \cap \overset{H_k(X)}{a}}$ $\in H_{n-k}(M)$

$\varphi : \mathrm{Hom}(H^{n-k}(M), \mathbb{F}) \to H_{n-k}(M)$

so $\boxed{\langle b, PD(a) \rangle = (a, b) = \langle a \cup b, [M] \rangle}$

a is originally upper K

PD sends a to lower $n-K$.

Geometric Poincare dual (little pd)

Thm. Property about map $H_n(M) \to H_n(M|x)$

If M is a connected n- manifold, the map

$$H_n(M) \to H_n(M|x) = H_n(M, M-x) \cong \mathbb{F} \text{ is injective.}$$

(admits fundamental class)

so, $[M]$ oriented $\Rightarrow H_n(M) \cong \mathbb{F} \underset{\text{UCT}}{\Rightarrow} H^n(M) \cong \mathbb{F} = \langle [M^*] \rangle$ where M^* is defined s.t. $\underbrace{\langle [M]^*, [M] \rangle}_{\text{the upper-lower inner prod.}} = 1 \in \mathbb{F}$.

Proof : See moodle

Remark. Some properties about submanifolds

Assume $i : N \hookrightarrow M$ is a smooth, closed, connected, $\mathbb{F}$-oriented submanifold.

let V be a Tubular nbd of N. Then

V is a subset of M. $(V, V-N)$ $V^{\#}$ is V minus N which is the 0-section.

$$\begin{tikzcd}
(M, \emptyset) \arrow[r, "j"] \arrow[dr, "j_x"'] & (M|N) \arrow[d] & (V|N) \arrow[l] \arrow[r, phantom, "\cong"] & (V, V^{\#}) \arrow[ll, bend left, "i"] \\
 & (M|x) & &
\end{tikzcd}$$

By this thm

N connected $\Rightarrow H^k(N) \cong \mathbb{F} = \langle [N]^* \rangle$

$\Rightarrow H^n(V, V^{\#}) \cong \mathbb{F} = \langle u \cup \pi^* [N]^* \rangle$ u is an orientation for $V_{M/N}$, (Thom iso).

$\Rightarrow H_n(V, V^{\#}) \cong \mathbb{F}$

Now, $i_* : H_n(V, V^{\#}) \xrightarrow{\cong} H_n(\overset{(M,\, M-N)}{M|N}) \cong \mathbb{F}$ (Excision)

$j_* : H_n(M) \xrightarrow{\cong} H_n(\overset{(M,\, M-x)}{M|x}) \cong \mathbb{F} \Rightarrow j_* : H_n(M) \xrightarrow{\cong} H_n(M|N)$ Some remark?

$\Rightarrow i_*^{-1} j_* [M]$ generates $H_n(V, V^{\#}) \cong \mathbb{F}$.

$\Rightarrow \langle \underbrace{u \cup \pi^* [N]^*}_{H^n(V, V^{\#})}, \underbrace{i_*^{-1} j_* [M]}_{H_n(V, V^{\#})} \rangle = k \in \mathbb{F}^*$

this thing divide by k

def. Orientation on $V_{M/N}$

$u_{M/N} = k^{-1} u$ is the orientation on $V_{M/N}$ induced by $[N]$ and $[M]$.

It satisfies $\boxed{\langle u_{M/N} \cup \pi^* [N]^*, \ i_*^{-1} j_* [M] \rangle = 1}$

def Geometric Poincare dual (little Pd)

$\mathrm{pd}(N) = j^*((i^*)^{-1}(U_{M|N})) \in H^{n-k}(M)$

$$\boxed{\begin{matrix} i^{*-1}: H^{n-k}(V, V^\#) \longrightarrow H^{n-k}(M|N) \\ j^*: H^{n-k}(M|N) \longrightarrow H^{n-k}(M) \end{matrix}}$$

Prop. Combining PD with pd.

If $a \in H^k(M)$, $\boxed{\langle \overset{H^{n-k}(M)}{\overset{\downarrow}{\mathrm{pd}(N)}} \cup a, \overset{H_n(M)}{\overset{\downarrow}{[M]}} \rangle = \langle \overset{H^k(M)}{\overset{\downarrow}{a}}, \overset{H_k(M)}{\overset{\downarrow}{i_*[N]}} \rangle}$

$i: N \hookrightarrow M$
$i_*[N] \in H_k(M)$

i.e. $\mathrm{PD}(\mathrm{pd}(N)) = i_*[N]$ as $\langle a, \mathrm{PD}(\mathrm{pd}(N))\rangle = \langle \mathrm{pd}(N) \cup a, [M]\rangle$

lemma: let $i: \overset{\text{Tubular nbd of } N.}{\overset{\downarrow}{V}} \to M$, then $i^*(a) = \langle a, i_*[N]\rangle \pi^*[N]^*$

Proof: $\pi: V \to N$ is an $\sim$ equivalence. $H^*(V)$ is generated by $\pi^*[N]^*$.

So it's enough to check that

$\langle i^*(a), [N]\rangle = \langle\langle a, i_*[N]\rangle \pi^*[N]^*, [N]\rangle$ Why?

this is an exercise. ■

Proof of prop:

If $b \in H^l(M|N)$, $j^*(b \cup a) = j^*(b) \cup a$ $j^*(a) = a$ as $a \in H^*(M)$

So $\langle \mathrm{pd}(N) \cup a, [M]\rangle = \langle j^*((i^*)^{-1}(U_{M|N})) \cup a, [M]\rangle$

$= \langle (i^*)^{-1}(U_{M|N}) \cup a, j_*[M]\rangle$

$= \langle (i^*)^{-1}((U_{M|N}) \cup i^* a), j_*[M]\rangle$

$= \langle U_{M|N} \cup i^*(a), i_*^{-1}(j_*[M])\rangle$ move $U_{M|N}$ to $\cap$ right

apply lemma

lemma $= \langle U_{M|N} \cup \langle a, i_*[N]\rangle \pi^*[N]^*, i_*^{-1} j_*[M]\rangle$ move $U_{M|N}$ back to left.

$= \langle a, i_*[N]\rangle$ defn of geometric pd. ■

Week 8 lec 4 lec 24.

5.2 Finish. have coeff in $\mathbb{F}$.

Recall: M closed connected, $\mathbb{F}$-oriented n-manifold.

PD: $H^k(M) \longrightarrow H_{n-k}(M)$

$\boxed{\langle b, \mathrm{PD}(a)\rangle = (a, b) = \langle a \cup b, [M]\rangle}$

$N \hookrightarrow M \longrightarrow \mathrm{pd}(N) \in H^{n-k}(M)$

$\langle \mathrm{pd}(N) \cup a, [M]\rangle = \langle a, i_*[M]\rangle = \mathrm{PD}(\mathrm{pd}(N)) = [N]$

Consider $\Delta : M \longrightarrow M\times M$

$x \mapsto x \times x$

show PD is $\cong$ by considering $pd(\Delta) \in H^n(M\times M)$

i.e. Δ is a submanifold of deg n. $M\times M$ is of $2n$.

$pd(\Delta) \in H^n(M\times M)$ and $PD: H^n(M\times M) \to H_n(M\times M)$

Homology of products ($\mathbb{F}$ coefficients)

$$\mathrm{Hom}(A\otimes B, \mathbb{F}) \cong \mathrm{Hom}(A,\mathbb{F}) \otimes \mathrm{Hom}(B,\mathbb{F})$$

so,

$$H_*(X\times Y) \cong \mathrm{Hom}(H^*(X\times Y), \mathbb{F})$$

By exterior product.

$$\cong \mathrm{Hom}(H^*(X)\otimes H^*(Y), \mathbb{F}) \cong \mathrm{Hom}(H^*(X),\mathbb{F}) \otimes \mathrm{Hom}(H^*(Y),\mathbb{F})$$

$$= H_*(X) \otimes H_*(Y)$$

$$\begin{tikzcd} H_*(X\times Y) \arrow[d] & \\ \alpha\times\beta \arrow[r, dash] & \alpha\otimes\beta \end{tikzcd}$$

denote $\alpha\times\beta$ the element corresponding to $\alpha\otimes\beta$ under this iso.

so, equivalently, the following can be characterized.

$$\boxed{\langle a\times b, \alpha\times\beta\rangle = \langle a,\alpha\rangle\langle b,\beta\rangle}$$

$\uparrow$

this is exterior prod of $H^*(X)$, $H^*(Y)$.

Recall $\langle b, z\cap a\rangle = \langle a\cup b, z\rangle$

?

lemma 1. $(z_1\times z_2)\cap(a_1\times a_2) = (-1)^{|a_2|(|z_1|-|a_1|)}(z_1\cap a_1)\times(z_2\cap a_2)$

Proof: check $\langle b_1\times b_2, LHS\rangle = \langle b_1\times b_2, RHS\rangle$.

$\langle b_1\times b_2, (z_1\times z_2)\cap(a_1\times a_2)\rangle$

$= \langle a_1\times a_2 \cup b_1\times b_2, z_1\times z_2\rangle$

$= \langle(-1)^{|a_2||b_1|}(a_1\cup b_1)\times(a_2\cup b_2), z_1\times z_2\rangle$

$\langle b_1\times b_2, RHS\rangle$

$= \langle b_1, z_1\cap a_1\rangle\langle b_2, z_2\cap a_2\rangle$

$= \langle a_1\cup b_1, z_1\rangle\langle a_2\cup b_2, z_2\rangle$

lemma 2 If X is path connected, $p\in X$, so $H^0(X) \in [P]$ and $a\in H^k(X)$, $\alpha\in H_k(X)$

then $\boxed{\alpha\cap a = \langle a,\alpha\rangle[p]}$

Proof: $\langle 1, \alpha\cap a\rangle = \langle a\cup 1, \alpha\rangle = \langle a,\alpha\rangle$ and $\langle 1, [p]\rangle = 1$

$\langle 1, \langle a,\alpha\rangle[p]\rangle = \langle a,\alpha\rangle$

Both equal to $\langle a,\alpha\rangle$ so $\alpha\cap a = \langle a,\alpha\rangle[p]$.

lemma 3 $\Delta^*(a\times b) = a\cup b$

$\Delta : X \to X\times X$ $\quad a, b \in H^*(X)$ $\quad$ note $\pi_1\circ\Delta = \pi_2\circ\Delta = \mathrm{Id}_X$

$\Delta^*(a\times b) = \Delta^*(\pi_1^*(a)\cup\pi_2^*(b)) = \Delta^*\pi_1^*(a) \cup \Delta^*\pi_2^*(b) = a\cup b$

orient $M\times M$ by $\underbrace{[M\times M]}_{\text{fund class of } M\times M.} = [M]\times[M]$

let $\tilde{u} = pd(\Delta) \in H^n(M\times M)$ $\quad$ since Δ is an n-dim'l submfld of $M\times M$.

Prop 1. $\langle \tilde{u}, [M]\times[p]\rangle = (-1)^n$

Proof:

$\langle \tilde{u} \cup (1\times[M]^*), [M]\times[M]\rangle$

$= (-1)^n \langle (1\times[M]^* \cup \tilde{u}, [M]\times[M]\rangle$

$= (-1)^n \langle \tilde{u}, [M]\times[M] \cap (1\times[M]^*)\rangle$

$= (-1)^n \langle \tilde{u}, ([M]\cap 1)\times([M]\cap[M]^*)\rangle$

$\leftarrow$ this implication?

$= (-1)^n \langle \tilde{u}, [M]\times[p]\rangle$

$\uparrow$ $H_{\ell+k} \to H_k$ in 1st component $\Rightarrow$ deg 0 so $[p]$

on the other hand, $\tilde{u} = pd(\Delta)$

$\langle \tilde{u} \cup (1\times[M]^*), [M]\times[M]\rangle$

$= \langle pd(\Delta) \cup (1\times[M]^*), [M]\times[M]\rangle$

$= \langle 1\times[M]^*, i_*([M]\times[M])\rangle$

using identity $\langle pd(N)\cup a, [M]\rangle = \langle a, i_*[M]\rangle$

$= \langle 1\times[M]^*, [\Delta]\rangle$

This implication! ☆

$= \langle \pi_1^*(1)\cup\pi_2^*([M]^*), \Delta_*[M]\rangle$

$= \langle \pi_2^*([M]^*), \Delta_*[M]\rangle$

$$\boxed{\langle a, f_*(x)\rangle = \langle f^*(a), x\rangle}$$

$= \langle [M]^*, \pi_{2*}(\Delta_*[M])\rangle = \langle [M]^*, [M]\rangle = 1$ ▨

Prop 2. (important & interesting)

$\tilde{u}$ called the symmetrimizer

$\tilde{u} \cup (a\times b) = (-1)^{|b||a|}\, \tilde{u} \cup (b\times a)$

Proof: $V=$ Tubular nbd of Δ in $M\times M$.

$\pi: V\to\Delta$ is proj in normal bundle.

π, j_Δ are homotopy inverses.

$$\begin{tikzcd} & M \arrow[dl, "j_\Delta"'] \arrow[d, "\Delta"] \\ V \arrow[r, "i'"] \arrow[d, "j'"'] & M\times M \arrow[d, "j"] \\ (V|\Delta) \arrow[r, "i"'] & (M\times M|\Delta) \end{tikzcd}$$

$a\times b\in H^*(M\times M)$

$$\tilde{u}\cup(i')^*(a\times b)$$
$$=\tilde{u}\cup\underbrace{\pi^* j_\Delta^*}_{\text{identity}} i'^*(a\times b)$$
$$=\tilde{u}\cup\pi^*\Delta^*(a\times b)$$
$$=\tilde{u}\cup\pi^*(a\cup b) \quad \text{(lemma 3)}$$
$$=(-1)^{|a||b|}\tilde{u}\cup\pi^*(b\cup a)$$
$$=(-1)^{|a||b|}\tilde{u}\cup(i')^*(b\times a)$$

$(i')^*(b\times a)$
$=(i')^*[\pi_1^*(b)\cup\pi_2^*(a)]$
$=(i')^*(\pi^*(b\cup a))$

Apply $j^*(i^*)^{-1}$ to both sides give the result. ?

∎

Prop 3 $\langle\tilde{u}, PD(a)\times y\rangle=(-1)^{n(n-|a|)}\langle a,y\rangle$ $a\in H^K(M)$, $y\in H_K(M)$

Proof $\langle\tilde{u}, PD(a)\times y\rangle$

$=\langle\tilde{u},([M]\cap a)\times(y\cap 1)\rangle$

$PD(a)=[M]\cap a$ by defn.

1 is capping id

$=(-1)^0\langle\tilde{u},([M]\times y)\cap(a\times 1)\rangle$ By cap distributivity?

$=\langle(a\times 1)\cup\tilde{u},[M]\times y\rangle$

$=\langle(1\times a)\cup\tilde{u},[M]\times y\rangle$ prop 2 as is grading is 0

$=\langle\tilde{u},[M]\times y\cap(1\times a)\rangle$

$=(-1)^{n|a|}\langle\tilde{u},([M]\cap 1)\times(y\cap a)\rangle$ lemma1 $|[M]|-|y|=n$?

$=(-1)^{n|a|}\langle\tilde{u},([M]\cap 1)\times\langle a,y\rangle[p]\rangle$ identity $\alpha\cap a=\langle a,\alpha\rangle[p]$

$=(-1)^{n|a|}\langle\tilde{u},[M]\times[p]\rangle\langle a,y\rangle$

$=(-1)^{n|a|}\cdot(-1)^n\langle a,y\rangle$ prop 1

$=(-1)^{\boxed{n(|a|+1)}}\langle a,y\rangle$ same parity $n(n-|a|)$

∎

Thm. PD is $\cong$ (Recall PD: $H^K\to H_{n-K}$)

For $0\neq a\in H^K(M)$, choose $y\in H_K(M)$ s.t. $\langle a,y\rangle\neq 0$. Then, prop 3 $\Rightarrow$ $PD(a)\times Y\neq 0$. therefore $PD(a)\neq 0$. $\Rightarrow$ PD is injective. $\Rightarrow$ $\dim(H_*(M))=\dim(H^*(M))$ So PD is an $\cong$.

$PD: \bigoplus_i H^i(M)\longrightarrow\bigoplus_i H_i(M)$
$H^K(M)\longrightarrow H_{n-K}(M)$
} same dim so must be $\cong$

∎

intersection pairing $(a,b) = \langle a \cup b, [M] \rangle$

Cor $(\cdot,\cdot)$ is nondegenerate

if $0 \neq a \in H^k(M)$, $\exists b \in H^{n-k}(M)$ with $(a,b) \neq 0$.

Proof Do it yourself

Remark: an example of PD.

If $\{a_i\}$ is a basis for $H^*(M)$, let $\{b_i\}$ be the dual basis w.r.t. $\underset{\text{intersection pairing}}{(\cdot,\cdot)}$. i.e. $(a_i, b_j) = \delta_{i,j}$.

then, $\langle b_j, PD(a_i)\rangle = (a_i, b_j) = \delta_{ij} \Rightarrow \boxed{PD(a_i) = b_i^*}$ (dual basis w.r.t. $\langle \cdot,\cdot \rangle$.)

$\langle a_i, PD(b_j)\rangle = (b_j, a_i) = (-1)^{|a_i||b_j|} \delta_{ij}$ $\quad \boxed{PD(b_j) = (-1)^{|a_j||b_j|} a_j^*}$

$= \langle b_j \cup a_i, [M] \rangle$ $\quad$ note: b_j^*, a_j^* are in H_*

Cor. $\underset{}{\overset{pd(\Delta) =}{}}\ \tilde{u} = \sum_i (-1)^{|a_i|} a_i \times b_i^*$

Proof: $\langle \tilde{u}, a_i^* \times b_j^* \rangle = (-1)^{|a_i||n-a_i|} \langle \tilde{u}, PD(b_i) \times PD(a_j) \rangle$

let $s = |a_i|(n - |a_i|) + n|a_i|$

$= (-1)^s \langle b_i, PD(a_j) \rangle$ $\quad$ identity $\langle \tilde{u}, PD(a) \times y \rangle = (-1)^{n(n-|a|)} \langle a, y \rangle$

$= (-1)^s (a_i, b_j) = (-1)^s \delta_{ij}$ $\quad \equiv |a_i|^2 \equiv |a_i| \mod(2)$

∎

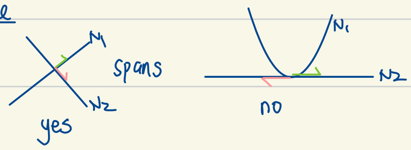
Intersection Pairing on homology

def. $N_1 \pitchfork N_2$

If $N_1, N_2 \hookrightarrow M$, are smooth submanis, N_1 is transverse to N_2 ($N_1 \pitchfork N_2$)

If $\underbrace{TN_1|_x}_{\text{tangent bundle of } N_1 \text{ at } x.} + TN_2|_x = TM|_x \quad \forall x \in N_1 \cap N_2.$

Example



doesn't span.

Prop. Properties if $N_1 \pitchfork N_2$:

1) $N_1 \cap N_2$ is a smooth submani of dim $\dim N_1 + \dim N_2 - \dim M$

2) $T(N_1 \cap N_2)|_x = TN_1|_x \cap TN_2|_x$

3) $V_{M/N_1 \cap N_2} = V_{M/N_1} \oplus V_{M/N_2}$ $\quad$ note $V_{M/N} = TN^{\perp} \subset TM|_N$ is the normal bundle and $TM|_N = V_{M/N} \oplus TN$

4) $\boxed{pd(N_1 \cap N_2) = pd(N_1) \cup pd(N_2)}$

def. $[N_1]\cdot[N_2]$ intersection pairing for smooth submani

$[N_1]\cdot[N_2] = (pd(N_1), pd(N_2))$

$= \langle pd(N_1) \cup pd(N_2), [M] \rangle$

when $N_1 \pitchfork N_2$, we have

$[N_1]\cdot[N_2] = \langle pd(N_1 \cap N_2), [M] \rangle$

$=$ # of points in $N_1 \cap N_2$ counted with intersection sign, if $\dim(N_1 \cap N_2) = 0$ and 0 o.w.

defn: j and i

$j : N_1 \hookrightarrow M$ be inclusion

$i = j|_{N_1 \cap N_2} : N_1 \cap N_2 \hookrightarrow N_2$

$pd(N_2) \in H^*(M)$, $j^* : H^*(M) \to H^*(N_1)$

Prop : $j^*(pd(N_2)) = pd_{N_1}(N_1 \cap N_2)$

Proof : $\underbrace{V_{N_1/(N_1 \cap N_2)} \cong i^* V_{M/N_2}}_{\text{By transverse.}}$, so $V_{N_1/N_1 \cap N_2} = j^* V_{M/N_2}$ V or u???

also why is $pd_{N_1}(N_1 \cap N_2)$ related $V_{N_1/N_1 \cap N_2}$?

Prop. $e(E)$

Spose $\pi : E \to M$ is an oriented VB.

$s : M \to E$ is a section, $s \pitchfork s_0$.

then $e(E) = pd_M(s \cap s_0) = pd_M(s^{-1}(0))$

Proof: $i_*^{-1}(U_E) = pd_E(s_0) = pd_E(s)$ since $s \sim s_0$.

Recall $pd(N) = j^*((i^*)^{-1}(U_{M/N}))$ and U_E is TC for E.

so $e(E) = s_0^*(i_*^{-1}(U_E)) = s_0^*(pd_E(s)) = pd_M(s_0 \cap s)$

cor: $\langle e(TM), [M] \rangle = \chi(M)$ (Euler characteristic)

Proof : in $M \times M$, $\Delta^* V_{M \times M/\Delta} \cong TM$?

so $\langle e(TM), [M] \rangle = [\Delta]\cdot[\Delta] = (\tilde{u}, \tilde{u}) \cong \chi(M)$ ▨

$$\boxed{\tilde{u} = \sum (-1)^{|a_i|} a_i \times b_i = \sum (-1)^{|b_i|} b_i \times a_i}$$