Alg Top

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1 Week 1

1.1 Lecture 1

Definition 1.1: Some conventions:

- $I = [0, 1]$
- $I^n = I \times I \times \ldots I$ for n times is the closed $n-$cube
- $D^n = \{\vec{v} \in \mathbb{R}^n \mid \|\vec{v}\| \leq 1\}$ is the closed $n-$dim disc
- $S^{n-1} = \{\vec{v} \in \mathbb{R}^n \mid \|\vec{v}\| = 1\}$
- $D^n \backsimeq I^n$, $S^{n-1} \subset D^n$, $D^n/S^{n-1} \backsimeq S^n$

Example 1.1 (Cylindrical coordinate):
You can consider transferring from spherical coordinate to cylindrical coordinate.

Definition 1.2:

1. Homotopic maps via a homotopy (the function that gives the homotopic maps).

Example 1.2 (Some homotopy maps):

1. $1_{\mathbb{R}^n}, 0_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ where 1 is the identity and 0 is the zero map, we have the homotopy $f_t(x) = tx$.
2. Consider the antipodal map, $A_n : S^n \to S^n, v \mapsto -v$. Note that $A_1 \sim 1_{S^1}$ via $f_t((z) = e^{i\pi t}z$. But is $A_2 \sim 1_{S^2}$? The answer is no.

Lemma 1.3:
Homotopy is an equivalence relation.

Definition 1.3: $[x, y] = Map(x, y)/\sim$, which are homotopy classes of maps $X \to Y$.

Lemma 1.4:
If $f_0, f_1 : X \to Y$ are homotopic via f_t and
$g_0, g_1 : Y \to Z$ are homotopic via g_t then
$g_0 \circ f_0 \sim g_1 \circ f_1$ via $g_t \circ f_t$.

Example 1.5 (**$[X, \mathbb{R}^n]$ has only 1 element.**)**:**
Let $f : X \to \mathbb{R}^n$. Then by the above lemma, and the fact that $1_{\mathbb{R}^n} \sim 0_{\mathbb{R}^n}$, we have

$$f = 1_{\mathbb{R}^n} \circ f \sim 0_{\mathbb{R}^n} \circ f = 0$$

This implies that $[X, \mathbb{R}]$ only has one element.

Definition 1.4 (Contractible space): A space X is contractible if the identity map is homotopic to some constant map C_p, which is a constant map for some $p \in X$. That is $1_X \sim C_p$.

Proposition 1.6 (Equivalent condition for contractible space):
A space Y is contractible if and only if $[X, Y]$ has only one element for all space X.

Proof:

- $\implies$: Suppose that Y is contractible. Then $1_Y \sim C_p$. Then let $f : X \to Y$ be any maps. Then $1_Y \circ f \sim C_p \circ f$ so $f \sim C_p$.
- $\impliedby$: Suppose that $[X, Y]$ has one element for all spaces X. Then Then all maps $f : X \to Y$ are homotopic to one another. Then $[Y, Y]$ only has one element. So $1_Y \sim C_p$.

$\square$

Remark that the definition of contractible is quite interesting. It is based on the quantity of the structure $[X, Y]$.

Definition 1.5 (Homotopy equivalent classes): Spaces X, Y are homotopic equivalent if there exists maps f, g, where $f : X \to Y$, $g : Y \to X$ such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$.
Examples of such include $\mathbb{R}^n, \{0\}$, and $\mathbb{R}^n \setminus 0 \sim S^{n-1}$.

Basic questions to motivate the study of algebraic topology

- Given the spaces X, Y, is $X \sim Y$?
- What is $[X, Y]$, i.e. the class of maps that goes from X to Y up to the homotopy of maps??

Definition 1.6 (Pairs of spaces): A pair of spaces (X, A) is a space $X, A \subset X$, and a map of pairs is $f(X, A) \to f(Y, B)$ such that $f : X \to Y$ is continuous, and $f(A) \subset B$.

Definition 1.7 (Homotopy of maps between maps of pairs): Let $f_0, f_1 : (X, A) \to (Y, B)$ be two maps of pairs. Then f_0, f_1 are homotopic if

$$f_0, f_1 : X \to Y \text{ are homotopic via } H : (X \times I, A \times I) \to (Y, B)$$

This means $(A \times I)$ never goes outside of B?

Remark:
If $f(X, A) \to (Y, B)$, $g : (Y, B) \to (G, C)$ are maps of pairs, so if $g \circ f$. We write $[(X, A), (Y, B)]$ for equivalence classes of maps of pairs.

Definition 1.8 (Homotopy groups $\pi_n(X, p)$): Let X be a space, and let $p \in X$, then the nth homotopy group is

$$\begin{aligned}
\pi_n(X, p) &= [(I^n, \partial I^n), (X, p)] \\
&= [(I^n, \partial I^n), (X, p)] \\
&= [(D^n, S^{n-1}), (X, p)] \\
&= [(s^n, *), (X, p)] \ni f
\end{aligned}$$

Note that $f \circ \alpha, \alpha = (D^n, S^{n-1}) \mapsto (D^n/S^{n-1}, S^{n-1}/S^{n-1}) \simeq (S^n, *)$.

There are multiple properties for the homotopy groups

1. There's a group structure. Note that $\pi_0(X, p)$ is the path components of X. Note that $\pi_1(X, p)$ is a group.(it is non-abelian and its abelianization is the homology group H_1.) $\pi_n(X, p)$ is an abelian group for $n > 1$. It has addition (abelian), identity map, and inverses.
2. Functoriality: if $f : (X, p) \to (Y, q)$, then it induces
$$f_* : \pi_n(X, p) \to \pi_n(Y, q)$$
$$[\psi] \mapsto [f \circ \psi]$$
Check that
$$(f \circ g)_* = f_* \circ g_*$$
3. Homotopy invariance:
If $f_0, f_1 : (X, p) \to (Y, q)$ have $f_0 \sim f_1$ as maps of pairs, then $f_{0*} = f_{1*}$ since
$$f_{0*}([\psi]) = ([f_0 \circ \psi]) = [f_1 \circ \psi] = f_{1*}([\psi])$$
Note that something interesting: point-based maps are automatically defined by the maps of pairs.
So
$$\pi_1(S^n, *) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & o.w. \end{cases}$$

But $\pi_i(S^n, *)$ is complicated in general.

n	$\pi_n(S^2)$
1	0
2	$\mathbb{Z}$
3	$\mathbb{Z}$
4	$\mathbb{Z}/2$
5	$\mathbb{Z}/2$
6	$\mathbb{Z}/12$
...	$\mathbb{Z}/3$, $\mathbb{Z}/15$,..
..	...

1.2 Lecture 2.

1.3 Singular Homology

1.3.1 Chain complex

Definition 1.9 (n-simplex): The $n-$simplex is $\triangle^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \text{all } t_i \geq 0, \sum_i t_i = 1\}$

For example $n = 1$, get $[0, 1]$ and $n = 2$, get the triangle. $n = 3$, we get a tetrahedron.

Definition 1.10 (Faces): If $I \subset \{0, \dots, n\}$, $f_I = \{\vec{t} \in \triangle^b \mid t_i = 0 \text{ if } i \notin I\}$ is the ith face of $\triangle^n$. By notation, we write $I = i_0 i_1 \dots i_k$ where it's in an ascenidng order.

Definition 1.11 (Face maps): $F_I : \triangle^{|I|-1} \to f_I \subset \triangle^n$, $F_I(\vec{t}) = \vec{x}$ where $x_i = \begin{cases} 0 & i \in I \\ t_j & i = 1_j \end{cases}$. This is basically to include the lower dimension simplex in the higher dimension one. All these maps are homeo and that for example,

$$F_{12} : \triangle^1 \to f_{12}, (t_0, t_1) \mapsto (0, t_0, t_1)$$

$$F_{02} : (t_0, t_1) \mapsto t_0, 0, t_1$$

Definition 1.12 (Chain complexes): Let R be a commutative ring. We can have $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}/n\mathbb{Z}$. Then a chain complex (C, d) over R is

1. $R-$modules $C_i, i \in \mathbb{Z}$, $C = \bigoplus_{i \in \mathbb{Z}} C_i$
2. $R-$linear maps $d_i : C_i \to C_{i-1}$, $d = \bigoplus d_i$, where $d : C \to C, d(C_i) \subset C_{i-1}$
3. $d_i \circ d_{i+1} : C_{i+1} \to C_{i-1}$ is 0 for all i, that is, $d \circ d = 0$ or $d^2 = 0$.

One tip is try to avoid subscripts if you can.

Definition 1.13 (homology group): If (C, d) is a chain complex over R, then the ith homology group is

$H_i(C) = \frac{\ker(d_i)}{\text{im}(d_{i+1})}$ an R-module. Note that people like to say the homology group, quite a lot, but it is more than a group. Note that $H_*(C) = \bigoplus_i H_i(C)$.

Some terminologies here:

- d is the differential/boundary map in the chain group
- $x \in \ker(d)$, x is closed / a cycle
- $x \in \text{im}(d_{i+1})$ x is exact / a boundary
- If $dx = 0$, write $[x]$ to be its image in $H_*(C)$.

Definition 1.14 (Chain complex of the n simplex):
The chain complex of the $n-$simplex is $(S_*(\triangle^n), d)$ where

$$S_k(\triangle^n) = \langle f_I \mid |I|_{I\subset\{0,\ldots,k\}} = k+1\rangle$$

The $k+1$ dimensional faces. The free $\mathbb{Z}$-module generated by $k-$dimensional faces of $\triangle^n$ for $k \geq 0$. For $k < 0$, $S_k(\triangle^n) = 0$, where $d(f_I) = \sum_j (-1)^j f_{I\setminus i_j}, I = i_0, \ldots, i_k$.

Example 1.7:
for $n = 2$,

$$d(f_{012}) = f_{12} - f_{02} + f_{01}$$

$$d^2(f_{012}) = (f_2 - f_1) + (f_2 - f_0) + (f_1 - f_0) = 0$$

Since that this is a freely generated abelian group, all ouu need to check is that d^2 maps all the basic elements to 0. So it is enough to check that $d^2 = 0$ on the generating elements f_I as f_I is a basis.

Proposition 1.8 (d^2=0):
Note

$$d^2(f_I) = \sum_{n_{jj'}} f_{I\setminus\{i_j, i'_j\}}, j < j'$$

where $n_{jj'} = (-1)^j(-1)^{j-1} + (-1)^{j'}(-1)^j = 0$ for the first one, we throw out i_j then $i_{j'}$ for the second one we throw out $i_{j'}$ then i_j.

Example 1.9 (Tetrahedron example):
$n = 2$, $\ker(d_2) = 0$, $im(d_3) = 0$.
we have $S_2(\triangle^2) = \langle f_{012} \rangle$, $S_1(\triangle^2) = \langle f_{01}, f_{12}, f_{02} \rangle$ and $S_0(\triangle^2) = \langle f_0, f_1, f_2 \rangle$.
we have

1. $\ker d_1 = im d_2 = \langle f_{12} - f_{02} + f_{01} \rangle$.
2. $\ker d_0 = \langle f_0, f_1, f_2 \rangle$
3. $\operatorname{im} d_1 = \{\sum a_i f_i \mid \sum a_i = 0\}$. it is generated by $\langle f_0 - f_1, f_1 - f_2, f_2 - f_0 \rangle$
4. so $\ker d_0 / \operatorname{im} d_1 \cong \mathbb{Z}$ by $\sum a_i f_i \to \sum a_i$
5. Therefore
$$H_i(S_*(\triangle^2)) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$
6. In fact
$$H_i(S_*(\triangle^n)) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Example 1.10 (Reduced chain):
The reduced chain C_X of $\triangle^N$ is $(\tilde{S_*}(\triangle^n), d)$ where

1. $\tilde{S_k}(\triangle^n) = S_k(\triangle^n), k \neq -1$
2. $\tilde{S_{-1}}(\triangle^n) = \langle f_\varnothing \rangle$ if $|I| = 1$, $df_I = f_\varnothing$
3. $df_\varnothing = 0$

It is an exercise to check $H_i(\tilde{S_*}(\triangle^2)) = 0, \forall i$. This is the most boring homotopy possible.

So the reduced simplicial homology is basically all the same except for the -1 index.

Definition 1.15 (Singular chain complex):
If X is a space, then its singular chain complex is $(C_*(X), d)$ where $C_k(X) = \{\sigma \mid \sigma : \Delta^k \to X \text{ is cts}\}$, is the free $\mathbb{Z}-$module generated by all maps of the form $\sigma : \Delta^k \to X$. This is really really big as it already has an uncountably large basis, which are formed by σs!

The elements of C_k are finite sums $\sum a_i \sigma_i, a_i \in \mathbb{Z}, \sigma_i : \Delta^k \to X$. $x \in C_k(X)$ is a singular chain. $\sigma : \Delta^k \to X$ is a singular simplex.

Definition 1.16 (The d map): Again since σ is a basis, just need to define $d(\sigma)$ and the rest follows from linearity.
If $\sigma : \Delta^k \to X$, then
$$d(\sigma) = \sum_{j=0}^{k} \sigma \circ F_{\hat{j}}$$
where $F_{\hat{j}} : \Delta^{k-1} \to \Delta^k$ is the face map.

Note that this is chosen such that

$$\phi_\sigma : S_*(\Delta^k) \to C_*(X)$$
$$f_I \mapsto \sigma \circ F_I$$

where $F_I : \Delta^{|I|-1} \to \Delta^k$ is a face map.
This satisfies $d \circ \phi_\sigma = \phi_\sigma \circ d$. In particular, we have $\phi = \sigma_\phi(f_{0..n})$ so that $d^2\sigma - 0$

Definition 1.17: Define $H_i(X) = H_i(C_*(X))$ to be the ith singular homology on X. What is the topology on X being used? it is fromt he continuous map $\Delta^n \to X$. Note that this is quite hard to be computed directly. So we use tools.

Example 1.11 ($X = \{\cdot\}$):
Consider the set $\{\cdot\}$ each $C_k(X) = \langle \sigma_k \rangle$. $\sigma_k : \Delta^k \to \{\cdot\}$ is the only map. so

$$d(\sigma_K) = \sum_{j=0}^{k} (-1)^k \sigma_{k-1} = \begin{cases} \sigma_{k-1} & \text{k even, } > 0 \\ 0 & \text{k odd} \end{cases}$$

So $\ker(d) = \langle \sigma_0, \sigma_1, \sigma_3, \sigma_5, \ldots \rangle$ and $\mathrm{im}(d) = \langle \sigma_1, \sigma_3, \sigma_5, \ldots \rangle$

So $H_*(C_*(X)) = \frac{\langle \sigma_0, \sigma_{odd} \rangle}{\langle \sigma_{odd} \rangle} = \mathbb{Z} = \langle [\sigma_0] \rangle$.
Hence

$$H_*(\{\cdot\}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & o.w. \end{cases}$$

2 Lecture 3

Recall with reduced homology,

$$\tilde{C_k}((x) = \begin{cases} C_k(X) & k \neq -1 \\ \langle \sigma_\phi \rangle & k = -1 \end{cases}$$

with $d\sigma = \sigma_\phi$ if $\sigma : \Delta^0 \to X$, $d\sigma_\phi = 0$
For exercise, check homology $\tilde{H}_i(\{\bullet\}) = 0, \forall i$.

Example 2.1:
Some examples are given, but too much to type down.

Proposition 2.2 (Path connected components):
If X is path connected, then $H_0 \cong \mathbb{Z} = \langle \sigma_p \rangle$ for any $p \in X$.

2.1 Subcomplexes, quotient complexes, and direct sum

Definition 2.1: If (C, d) is a chain complex over R, then a subcomplex of (C, d) is

- $A_i \subset C_i$ are submodules such that
- $d(A_i) \subset A_{i-1}$ so if $A = \bigoplus A_i$ then $d(A) \subset A$.

If (A, d) is a subcomplex of (C, d), then

- (A, d) is a chain complex
- $(C/A, d)$ is a chain complex where $C/A = \bigoplus_i C_i/A_i$

Note that $d_i(A_i) \subset A_{i-1}$ so d_i extends to $d_i : C_i/A_i \to C_{i-1}/A_{i-1}$ where $(C/A, d)$ is quotient complex.

Note that if $A \subset X$ then $C_*(A)$ is a subcomplex of $C_*(X)$.

Definition 2.2 (Chain complex for pair spaces): If (X, A) is a pair of spaces, let $C_*(X, A) = C_*(X)/C_*(A)$ is the singular chain complex of (X, A).

Note: If (C_α, d_α) $\alpha \in A$, are chain complexes, so is $(\bigoplus_{\alpha \in A} C_\alpha, \bigoplus_{\alpha \in A} d_\alpha)$. For example, $H_*(\bigoplus C_\alpha) = \bigoplus_{\alpha \in A} H_*(C_\alpha)$

Proposition 2.3:
$H_*(X) = \bigoplus_{X_\alpha} H_\alpha(X_\alpha)$ where the X_α are the path components of X.

Proof: Δ^k is connected so $Map(\Delta^k, X) = \coprod_\alpha Map(\Delta^k, X_\alpha)$ so $C_k((X) = \bigoplus_\alpha C_k(X_\alpha)$. This decomposition respects d so we have a direct sum of chain complexes.

□

2.2 Functors and induced maps

Definition 2.3 (Category): A category is:

- A collection of objects
- for each pairs of objects A, B, a set of morphisms $f : A \to B$ equipped with a composition rule: $f : A \to B$, $g : B \to C$ determines $g \circ f : A \to C$

satisfying

- $h \circ (g \circ f) = (h \circ g) \circ f$
- For each object A, $\exists 1_A : A \to A$ such that $f : A \to B$ $f \circ 1_A = f, 1_B \circ f = f$.

Example 2.4:
Write

$$\begin{cases} \text{objects} \\ \text{morphisms} \end{cases}$$

, we have

$$\begin{cases} \text{R-modules} \\ \text{R-linear maps} \end{cases} \begin{cases} \text{Top spaces} \\ \text{Continuous maps} \end{cases} \begin{cases} \text{pairs of spaces } (X, A) \\ \text{maps of pairs} \end{cases}$$

Definition 2.4 (Functors): If C_1, C_2 are categories, a functor $F : C_1 \to C_2$ assigns an object $a \in C_1$ to object $F(a)$, in C_2. For morphisms $f : A \to B$, it also gives $F(f) : F(A) \to F(B)$ satisfying

- $F(1_A) = 1_{F(A)}$
- $F(f \circ g) = F(g) \circ F(g)$

Definition 2.5 (Chain maps): Given chain complexes $(C, d), (C', d')$, chain complexes over a ring R, then a chain map is a function that respects the linearity and subset, with $d'f = fd$, or $d'_i \circ f_i = f_{i-1} \circ d_i$.

Lemma 2.5:
Identity map $1_C : (C, d) \to (C, d)$ is a chain map. Also composition of chain maps is chain map. Now we get category

$$\begin{cases} \text{Chain complexes over } R \\ \text{chain maps} \end{cases}$$

Lemma 2.6 (Well defined-ness):
If $f : (C, d) \to (C', d')$ is a chain map, then we can write $f_* : H_*(C) \to H_*(C')$ where $f_*([x]) = [f(x)]$. Call f_* to be the map induced by f.

Proof: fill in later □

Lemma 2.7:

- $(id_C)_* = id_{H_*(C)}$
- $(g \circ f)* = g_* \circ f_*$.

So homology defines a functor

$$\left\{\begin{array}{l}\text{Chain CX over } R \\ \text{Chain maps}\end{array}\right. \xrightarrow{H_*} \left\{\begin{array}{l}\text{R-modules} \\ \text{R-linear maps}\end{array}\right.$$

$$(C, d) \mapsto H_*(C)$$

$$f : C \to C' \mapsto f_* : H_*(C) \to H_*(C')$$

Definition 2.6 (#): Let $f : X \to Y$ be continuous maps, then define

$$f_\# : C_*(X) \to C_*(Y)$$

$$\sigma \in Map(\Delta^k, X) \mapsto f \circ \sigma$$

Lemma 2.8:
is a chain map

Proof: fill it in later. The main idea is that f is left composition and face map is right composition. □

There's a functor

$$\left\{\begin{array}{l}\text{Spaces} \\ \text{Continuous maps}\end{array}\right. \to \left\{\begin{array}{l}\text{Chain complexes over } \mathbb{Z} \\ \text{Chain maps}\end{array}\right.$$

$$X \mapsto C_*(X)$$

$$f : X \to Y \mapsto f_\# : C_*(X) \to C_*(Y)$$

Again, composition of functors is again a functor so

$$\left\{\begin{array}{l}\text{Spaces} \\ \text{Continuous maps}\end{array}\right. \to \left\{\begin{array}{l}\mathbb{Z}\text{-modules} \\ \mathbb{Z}\text{-linear maps}\end{array}\right.$$

$$X \mapsto H_*(X)$$

$$f : X \to Y \mapsto f_* : H_*(X) \to H_*(Y)$$

2.3 Week 2 lecture 1

Recall that $f_\#$ goes from $C_*(X)$ to $C_*(Y)$ and f_* is the one that goes from $H_*(X)$ to $H_*(Y)$.

Definition 2.7 (Maps of pairs): If $f : (X, A) \to (Y, B)$ then $f_\# : C_*(X) \to C_*(Y)$.
If $\sigma : \Delta^k \to A$ then $f \circ \sigma : \Delta^k \to B$ so it also contains in B So $f_\#(C_*((A))) \subset C_*(B)$.
Hence $f_\#$ descends to a map $C_*(X, A) \to C_*(Y, B)$.

As maps of pairs, we also get functors.

$$\begin{cases} \text{Pairs of spaces} \\ \text{Maps of pairs} \end{cases} \to \begin{cases} \text{Chain complexes over } \mathbb{Z} \\ \text{Chain maps} \end{cases} \to \begin{cases} \mathbb{Z}\text{-modules} \\ \mathbb{Z}\text{-linear maps} \end{cases}$$

$$(X, A) \mapsto C_*(X, A) = C_*(X)/C_*(A) \mapsto H_*(X, A)$$

$$(f : (X, A) \to (Y, B)) \mapsto (f_\# : C_*(X, A) \to C_*(Y, B)) \mapsto (H_*(X, A) \to H_*(Y, B))$$

Definition 2.8 (Chain homotopic definition): given $g_0, g_1 : C \to C'$ chain maps, then g_0 is homotopic to g_1 is there exist $h_i : C_i \to C'_{i+1}$ such that $d'h + hd = g_1 - g_0$.

Lemma 2.9:
Chain homotopy is equiv relation.

Proposition 2.10:
If $g_0, g_1 : C \to C'$ chain maps, $g_0 \sim g_1$ then $g_{1*} = g_{0*} : H_*(C) \to H_*(C')$.

Corollary 2.11:
$C \sim C'$ implies $H_*(C) \cong H_*(C')$.

Remark 1: There's lots of arguments on the idea of the proof, i.e. the chain homotopic arguments. This involves a confusing rectangular diagram, universal chain homotopy, with ψ mapping from $S_*(\delta^K) \to C_*(X)$. Also defined things such as convexity so we can dissect Δ^n and $\Delta^n \times I$s. Also lots of index drama.

Lemma 2.12 (Naturality):
The square involving $S_*(\Delta^k)$, $S_*(\Delta^n), C_*(\Delta^k \times I), C_*(\Delta^n \times I)$, commutes. After some long and complicated arguments, we get

Corollary 2.13:
$f_0 \sim f_1$ implies $f_{0*} \sim f_{1*}$.
Also $f : X \to Y, g : Y \to X$ induce homotopy equivalence. Also contractible has $H_*(X) = \mathbb{Z}$ with $* = 0$ and 0 otherwise.

2.4 Subdivision

Definition 2.9: Given sequence of R modules and linear maps, the defs for following:

- exact at A_i
- Note sequence is exact $\iff$ (A, f) is a chain complex with $H_*(A) = 0$
- $0 \to A \to 0$
- $0 \to A \to B \to 0$
- $0 \to A \to B \to C \to 0$
- SES

Remark 2 (A quite famous example of SES):

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{\pi} C_*(X)/C_*(A) \to 0$$

Theorem 2.14 (Snake lemma):
Turn a SES into a LES of the H_is.

Corollary 2.15:
If (X, A) is a pair of spaces then we have the LES

$$...H_{i+1}(X, A) \to H_i(A) \xrightarrow{i_*} H_i(C) \xrightarrow{\pi_*} H_i(X, A) \to H_i(A)....$$

where it's ∂ for the blank arrows.

Now we could use this to compute the LES of $(X, \{p\})$.
For $i > 0, H_i(X) \cong H_i(X, \{p\})$. but $H_0(X) = H_0(X, \{p\}) \oplus \mathbb{Z}$.

3 Week 2 lecture 3

Lemma 3.1:
$\tilde{H}_i(X) \sim H_i(X, p)$

Proof: The basic idea is to show $H_i(X, p)$ is the same in the reduced homology. Then use the inclusion to make an SES. Use snake lemma to extend it to LES. $\square$

Remark 3 (Subdivision lemma): You make an open cover U of X, and make $C_k^U(X)$. These are $\sigma^K \to X$ that are restrained in a particular triangle.

Then, we get a lemma called the subdivision lemma that allows you to do this: for U an open cover of X, you get

$$i_* = H_*^U(X) \to H_*(X)$$

is actually an isomorphism.

The proof idea here is to define barycentric subdivision on the set X until each of the triangles lie in a single element of open cover. THen you get some chain homotopic maps.

This gives you the Mayer Vietoris sequence as you get open cover U, V on X.

Proposition 3.2 (Diamond SES):
If the commutative diagram of inclusions is good, i.e. $j_2 \circ i_2, j_1 \circ i_1$, then there's a SES

$$0 \to C_*(U_1 \cap U_2) \xrightarrow{i} C_*(U_1) \oplus C_*(U_2) \xrightarrow{j} C_*^U(X) \to 0$$

where

$$i = \begin{bmatrix} i_{1\#} \\ i_{2\#} \end{bmatrix}$$

$$j = \begin{bmatrix} j_{1\#} & j_{2\#} \end{bmatrix}$$

Note that you have the C_*^U at the end because it's induced by an open cover.

Proof:
Idea is to check exactness in the three middle parts. i.e. i injective, j surjective, middle isomorphic.
The idea for the middle one being exact is: since digram commute then $\operatorname{im} i \subseteq \ker j$. For the other direction, say something is in $\ker j$, we construct the preimage of it as a sum. But they are the same after rearranging. So they must come be linear combinations of things coming from $U_1 \cap U_2$. So $\ker j \subseteq \operatorname{im} i$. □

Corollary 3.3 (Mayer Vietoris LES):
Recall from before.

Proposition 3.4 (Reduced hom of S^n):
$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

3.1 Week 3 lecture 1

Lemma 3.5 (Turning a commuting diagram of SES into LES):
Suppose we have a commuting diagram of chain complexes and chain maps, with rows being SES as below, then we can get a commuting diagram of LES as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
\bullet & \longrightarrow & H_i(B) & \longrightarrow & H_i(C) & \xrightarrow{\partial} & H_i(A) & \longrightarrow & \bullet \\
 & & \downarrow{\scriptstyle f_{B*}} & & \downarrow{\scriptstyle f_{C*}} & & \downarrow{\scriptstyle f_{A*}} & & \\
\bullet & \longrightarrow & H_i(B') & \longrightarrow & H_i(C') & \xrightarrow{\partial} & H_i(A') & \longrightarrow & \bullet
\end{array}$$

Proposition 3.6:
Let r_n be the antipodal map from $S^n \to S^n$. Then

$$r_{n*} : \tilde{H}_n(S^n) = \langle [S^n] \rangle \to \tilde{H}_n(S^n) = \langle [S^n] \rangle$$

is given by $r_{n*}[S^n] = -[S^n]$

Corollary 3.7 (Some theorem about flipping w.r.t. point):
not quite understanding the corollary and the proof.

3.2 Excision and collapsing on a pair

Definition 3.1 (Deformation retract): There's a really interesting way of defining deformation retract as maps of pairs.
Suppose $A \subset Z$, then A is deformation retract of Z if there is a map $p : (Z, A) \to (A, A)$ such that

$$p \circ i : (A, A) \to (A, A) = 1_{(A,A)}$$

$$i \circ p : (Z, A) \to (Z, A) \sim 1_{(Z,A)}$$

as a map of pairs. There is a way easier way to define deformati on retract but this one works.

Definition 3.2 (Good pair): A pair (X, A) is afood pair if $\exists U \subset X$ open such that $A \subset U$ and U deformation retracts onto A.

Theorem 3.8 (Excision theorem):
This theorem is good for example sheet
If (X, A) is a good pair, and $\pi : (X, A) \to (X/A, A/A)$. Then

$$\pi_* : H_*(X, A) \to H_*(X/A, A/A) \cong \tilde{H}(X/A)$$

is an isormorphism.

Remark 4 (Some use of the excision theorem):
In class, we used the theorem to compute the homology group of $Z = S^2/\{n, s\}$ via a bunch of sequences. We also used the theorem to compute the homology group of T/B where B is a circle, where $T/B = Z$.

3.3 Week 3 lecture 2

Proposition 3.9 (The five lemma):
The five lemma

If $U_{i \in J}$ forms a open cover of X, then for $A \subset X$, $U_{i \in J}$ by intersecting each element with A gives you an open cover of A. So $C_*^{U_A}(A) \subset C_*^U(X)$. We define $C_*^U(X, A) = C_*^U(X)/C_*^{U_A}(A)$, then we get this map

$$i : C_*^U(X) \to C_*(X)$$

induces

$$i : C_*^U(X, A) \to C_*(X, A)$$

This follows from subdivision lemma.

Lemma 3.10 (Subdivision lemma for pairs):

$$i_* : H_*^U(X, A) \to H_*(X, A)$$

is an isomorphism.

Proof:
Use snake and give on a commuting diagram of SES. ☐

Theorem 3.11 (Excision):
Suppose that $B \subset A \subset X$and $\overline{B} \subset \text{int}(A)$. Let

$$j : (X - B, A - B) \to (X, A)$$

be the inclusion. Then

$$j_* : H_*(X - B, A - B) \to H_*(X, A)$$

is an isomorphism.

Proof:
Split the open cover (C_*^U) into a direct sum, use a lemma that C_*^U isomorphism yields H_*^U isomorphism. Then use subdivision lemma and a commuting square to show isormopshism of H_*^Us.

□

Proposition 3.12 (LES of Triple Inclusion):
Suppose $Z \subset Y \subset X$ then

$$\mapsto H_*(Y, Z) \mapsto H_*(X, Z) \mapsto H_*(X, Y) \mapsto H_{*-1}(Y, Z) \mapsto$$

Where first two arrows are inclusions and the last one is the boundary given by snake.

Lemma 3.13:
If A is a d.r. of U and $U \subset X$, and if $j(X, A) \to (X, U)$ is the inclusion, then

$$j_* : H(X, A) \to H_*(X, U)$$

is an isomorphism.

Proof: The proof utilizes the L.E.S. of (U, A) and the LES of (X, U, A). □

Definition 3.3 (Good pair): (X, A) is a good pair if $\exists U \subset X$ open and that $A \subset U$ a d.r. of U and $\overline{A} \subset U$. Or say A is closed in X.

Theorem 3.14 (Collapsing of a pair):
Let (X, A) be a good pair, $\pi : (X, A) \to (X/A, A/A)$ be the quotient map. Then $\pi_* : H_*(X, A) \to H_*(X/A, A/A)$ is an isomorphism.

Proof: Commuting diagram □

Definition 3.4 ($n-$manifold): A space X is an n-manifold if it is metrizable. i.e. Hausdorff and second countable. And every $x \in X$ has open neighbourhood $U_X \cong \mathbb{R}^n$.

Proposition 3.15:
If X is an n manifold and $x \in X$ then

$$H_*(X, X - x) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.16:
If M^m, N^n are m, n manifolds and M, N homeomorphic, then $m = n$.

Proof: Fill it in. But question: how did you get from the $X, X-$ to D^n, $D^n - 0$? □

4 Week 3 lecture 3

Definition 4.1 (Degree of map): let $f : S^n \to S^n$ with $f_*[S^n] = k[S^n]$ wher k is the degree of f.

Proposition 4.1 (Properties about degree of maps):

1. $(1_{S^n})_* = 1_{H_*(S^n)}$ so $\deg id = 1$.
2. homotopic maps have the same degrees
3. composition of maps have their degrees being multiplicative.
4. If $f : S^n \to S^n$ is a homeomorphism then $\deg f = \pm 1$. It is orientation preserving if $\deg f = 1$, otherwise it is orientation reversing
5. if $r_* : S^n \to S^n$ is reflecting in $V^\perp$, then $\deg r_v = -1$
6. if $A : S^n \to S^n$ is antipodal map, then $A = r_{e_1} \circ \ldots \circ r_{e_{n+1}}$ implies $\deg(A) = (-1)^{n+1}$. Using properties 3 and 5.

Corollary 4.2:
$A \not\simeq 1_{S^n}$ if n is even.

4.1 Local degree

`https://math.stackexchange.com/questions/2205452/local-degree-of-a-map-between-n-spheres`
Given $p \in Such^n$, then $S^n - p \cong D^{oN}$ is contractible. So

$$\pi_* : H_n(S^n) \to H_n(S^n, S^n - p)$$

is an isomorphism for $n \geq 1$, as LES of a pair.

We now define $[S^n, S^n - p]$.

Definition 4.2 (We define a bunch of things to used to study what happens with on S^n): $p \in S^n$, get a open $U, p \in U$, define $B = S^n \setminus U$. Then B is closed. Then define a bunch of maps of pairs with respect to this convention.

We then generate a bunch of commutative diagrams. i.e. with $U' \subset U$, get $H_n(U', U' - p) \to H_n(U, U - p)$ is a homeomorphism.
Intuition: Now we think about local degree. Consider a loop $S^1 \to S^1$ that is homotopic to the constant loop but might go back and forth hovering over a point. The global map still have degree one but the preimage of one point could be a few points.

Definition 4.3 (Local degree): Consider $f : S^n \to S^n$ with $f^{-1}(p) = \{q_1, \ldots, q_r\}$ finite. By Hausdorffness of S^n, find $U_i \subset S^n$ open nbhds of qs pariwise disjoint.
We obtain maps $f_i : (U_i, U_i - q_i) \to (S^n, S^n - p)$.
So $f_*[U_i, U_i - q_i] \to k[S^n, S^n - p]$. (where the $[\cdot]$ is the generator. i.e. where does f_* send the generator to the generator of some subgroup of $\mathbb{Z}$). We then define $\deg_{q_i} f = k$ to be the local degree of f at q_i.

Lemma 4.3:
Note that this does not depend on the choice of U_i. The proof is again some relative homology commutative diagram.

Note that $V = \coprod U_i \subset S^n$, which is open. We can use excision to show that $[U_i, U_i - q_i]$ form a basis. So.

Lemma 4.4 (Another way to study map $H_n(S^n)$):
The map

$$H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p)) \cong \oplus H_n(U_i, U_i - q_i)$$

is given by

$$[S^n] \mapsto \sum_r [U_i, U_i - q_i]$$

Proof: By two complicated commutative diagrams. □

Theorem 4.5 (Degree of f computed as local degrees):
Suppose $f : S^n \to S^n$, $f^{-1}(p) = \{q_1, \ldots, q_r\}$ as above. Then $\deg f = \sum_{i=1}^r \deg_{q_i} f$. Note that this is true no matter which point p we pick.

Proof: Again a complicated commutative diagram. □

Example 4.6:
Two examples given. One given an example of degree map, and another demonstrates how you can compute $f_*[U_0, U_0 - 1]$ by where it sends a rotation of it.

5 Week 4

5.1 Lecture 1

Remark 5 (Harewicz homomorphism):
It is a result that link homotopy to homology.

$$\Phi : \pi_n(X, *) \to H_n(X)$$

$$f \mapsto f_*[S^n]$$

In general, this map is quite far away from an isomorphism.

$$h_* : \pi_n(X) \to H_n(X)$$

He gave me an example of where $H_2(T^2)$ is $\mathbb{Z}$, and that $f_*(S^2)$ mapped it to 0.
A better model is that if M is a closed, compact, connected n manifold. Then we will show $H_n(M) \cong \mathbb{Z} = \langle [M] \rangle$.

Remark 6:
So the prof give out two intuitions. First one is Harewicz homomorphism which is not an isomorphism from $\pi(X, *)$ to study H_n, so it is bad. Then he introduced the manifolds one, which would actually say that the genus 2 and genus 3 surfaces are the same, which is also not that good. Then he that's why he introduced **cellular CX**.

Definition 5.1 (attaching/gluing along function): Suppose $B \subset Y$, $f : B \to Y$, then $X \cup_f Y = (X \coprod Y)/\sim$. where $\sim$ is the smallest equivalence relation containing $b \sim f(b)$ $\forall b \in B$. This is the space obtained by gluing Y to X along f.why the smallest equivalence relation?Why not pointwise gluing?
If $(Y, B) = (D^k, S^{k-1})$ then $X \cup_f D^k$ is obtained by attaching a $k-$cell to X.

Definition 5.2 (Finite cell complex, skeletons): A finite cell complex (fcc) is a space equipped with closed sets

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n$$

Such that for each k, X is obtained by attaching finitely many cells to X_{k-1}. such that the following holds:

$$X_k \simeq X_{k-1} \bigcup_F \coprod_{\alpha \in A_k} D^k$$

$$F : \coprod_{a \in A} S^{k-1} \to X_{k-1}$$

$$F = \coprod f_\alpha : S^{k-1} \to X_{k-1}$$

where each small f_α is the
We could also drop the finiteness conditions. X_k is the k skeleton of X.

Definition 5.3 (Wedge product): For pointed space

$$\bigvee i \in I(X_i, x_i) = \coprod_{i \in I} X_i / \coprod_{i \in I} x_i$$

Example 5.1 (Projective spaces):
Consider the n dimensional projective space. In this section, he showed it is compact and hausdorff. Also defined the Hopf map, and the following CW construction for the $\mathbb{CP}^n$.

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - 0/\mathbb{C}^*$$

Proposition 5.2 (Construction for $\mathbb{CP}^n$):

$$\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \bigcup_{p_{n-1}} D^2 n$$

$$P_{n-1} : S^{2n-1} \to \mathbb{CP}^{n-1}$$

Theorem 5.3 (The homology classes for $\mathbb{CP}^n$):
as CW complexes constructions, also induction for the homology group construction.

6 Week 4 lecture 3

Observe that $X_k/X_{k-1} \simeq \bigvee_{\alpha\in A_k} S^k$.
So note that $H_k(X_k, X_{k-1}) \simeq H_k(\bigvee_{\alpha\in A_k} S^k) = \langle e_\alpha \mid \alpha \in A_k\rangle$.
We have a map from $\bigvee_{\alpha\in A_k} S^k$ mapping to S^k.
We can use the LES of triple on (X_k, X_{k-1}, X_{k-2}).

Lemma 6.1:
$d_k = \pi_{(k-1)*} \circ \partial_k$ where $\partial_k : H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1})$ is ∂ in the LES of (X_k, X_{k-1}).

Corollary 6.2:

$$d_k \circ d_{k+1} = 0$$

Definition 6.1 (Cellular chain complex): If X is a fcc, $H_*^{\text{cell}}(X) = H_*(C_*^{\text{cell}}(X)) \simeq H_*(X)$.

Theorem 6.3:

$$H_*^{\text{cell}}(X) = H_*(C_*^{\text{cell}}(X)) \simeq H_*(X)$$

Remark 7 (The cellular homology of $\mathbb{RP}^n$ and $\mathbb{CP}^n$):

Lemma 6.4:
Suppose X is a fcc with one $0-$cell, all other cells have $m \leq dim \leq M$,
then $\tilde{H}_*(X) = 0$ if $* < m, * > M$. The proof is by induction.

7 Week 4 lecture 3

Lemma 7.1:
If X is a fcc then (X, X_k) is a good pair.

Corollary 7.2:
If X is a fcc then $H_k(X_{k+1}) \simeq H_k(X)$. This is true for any k.

Theorem 7.3:
If X is a fcc then $H_*^{\text{cell}}(X) \simeq H_*(X)$.

Definition 7.1 (Tensor products):

Remark 8 (Tensor products and their properties): Three properties plus $\otimes M$ gives a functor. THis gives us that if (C, d) is a chain complex over R then $(C \otimes M, d \otimes 1)$ gives another chain complex.

Definition 7.2 (Singular chain complex with coefficients in a abelian group): If G is a $\mathbb{Z}$module, then $C_*(X; G) = C_*(X) \otimes G$ is the singular chain complex with coefficient in G. $H_*(X; G)$ is its homology.

Remark 9 (Euler characteristic): the definition, and the theorem regarding Euler characteristic.

Theorem 7.4 (Eilenberg Steenrod axioms):
Define an ordinary homology theory with coefficient in G. It is a functor from (Pairs of spaces, maps of pairs), ($\mathbb{Z}$-modules, $\mathbb{Z}$-linear maps.) satisfying four different axioms. Then if X is a fcc, and H_* is any functor satisfying the axioms, $H_*(X) \simeq H_*(C_*^{\text{cell}}(X) \otimes G)$. In particular, $H_*(X; G)$ satisfy the axioms.

8 Week 5

8.1 Week5 lecture 1

`https://people.math.osu.edu/broaddus.9/6802/files/lecture04.pdf`

Definition 8.1 (Free resolution): If R is a module then a free solution of M is a free chain CX A over R , each A_k is free over R, such that

- $A_k = 0$, $k < 0$
- $H_*(A) = \begin{cases} M & * = 0 \\ 0 & * \neq 0 \end{cases}$

It is better viewed this way (question, are this two defns equivalent?)

$$\ldots F_2 \to F_1 \to F_0 \to M \to 0$$

Each F are free and abelian.
Note that in here, if R is a PID, then many good things happen. However, if R is not a PID things are not as good. So algebraic geometry studies the situation when R is not a PID.

Definition 8.2 (Torsion): If M, N are modules, then $\text{Tor}_i(M, N) = H_i(A \otimes N)$ where A is a free resolution of M. Torsion is the measure of failure of $H_*(A \otimes N) = H_*(A) \otimes N$.
The above pdf gives a short list of properties of torsion.

Definition 8.3 (Short injective): A chain CX is short injective if

- C_* is 0 whenever $* \neq k, k+1$, and C_k, C_{k+1} are free over R.
- $d : C_{k+1} \to C_k$ is injective.

Note that it it not necessarily an exact one so it could be not injective.

Theorem 8.1 (A theorem analogues to the structure thm of f.g. module over PID):
A free CX over a PID is a direct sum of short injective chain complexes.
`https://en.wikipedia.org/wiki/Structure_theorem_for_finitely_generated_modules_over_a_principal_ideal_domain`

Corollary 8.2:
If two free chains of complexes over a PID have $\simeq$ homology then they are chain homotopic equivalent.

Corollary 8.3:
If C is a chain complex over a field $\mathbb{F}$ then $C \simeq (H_*(C), 0)$.

Corollary 8.4 (Universal coefficient theorem):
If C is a free CX over a PID, then

$$H_k(C \otimes N) = \mathrm{Tor}_0(H_k(C), N) \oplus \mathrm{Tor}_1(H_{k-1}(C), N).$$

Therefore as a result, $H_*(X, G)$ is determined by $H_*(X)$.
We need a PID so the 0th and 1th things are the only nonzero quantities. Then we can split them into things to study. That's why PID is important.

9 Cohomology and products

Yay, finally on cohomology

Definition 9.1: If M, N are $R-$modules, then $\mathrm{Hom}(M, N)$ is an R-module.
If $f : M_1 \to M_2$ are $R-$linear, then

$$f^* : \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$$

$$\alpha \mapsto \alpha \circ f$$

then f^* is an $R-$linear map.
Note that

$$(f \circ g)^*(\alpha) = g^*(f^*(\alpha))$$

This gives a contravariant functor. (see notes for more details).

Note that there is a covariant functor is a functor. A contravariant functor is not a functor.
If (C, d) is a chain complex over R, then

$$(\mathrm{Hom}(C, N), d^*) = \bigoplus_k \mathrm{Hom}(C_k, N)$$

where

$$d_k^* : \mathrm{Hom}(C_{k-1}, N) \to \mathrm{Hom}(C_k, N)$$

satisfies $(d^*)^2 = 0$. By the contravariant functor property.

Definition 9.2 (The cochain complex): So $(\mathrm{Hom}(C, N), d^*)$ is a cochain complex. d^* raises the homological degree by 1.

Then there is a covariant functor

$$\begin{cases} \text{Chain CXs over } R \\ \text{Chain maps} \end{cases} \to \begin{cases} \text{Cochain complexes} \\ \text{cochain maps} \end{cases}$$

$$(C, d) \mapsto (\mathrm{hom}(C, N), d^*)$$

$$f : C \to C' \mapsto f^* : \mathrm{Hom}(C', N) \to \mathrm{Hom}(C, N)$$

Definition 9.3 (Cohomology): Cohomology of (C^*, d_k^*) is

$$H^k(C) = \ker d_k^* / \operatorname{im} d_{k-1}^*$$

This also gives a contravariant function for the pairs of spaces.

Definition 9.4: Given a space X, define its singular cochain complex with coefficients in group G. The cochain complex is

$$(\mathrm{Hom}(C_*(X), G), d^k) = (C^*(X; G))$$

$$H^k(X; G) = H^k(C^*(X; G))$$

9.1 Week 5 Lec 2

Definition 9.5 (kth singular homology): Cohomology of (C^*, d_k^*) is

$$H^k(C; G) = \ker d_{k+1}^* / \operatorname{im} d_k^*$$

Definition 9.6 (Cochain maps): If $f : X \to Y$ then $f^\# : C^k(Y; G) \to C^k(Y; G)$
$f^\#(\alpha)(\sigma) = \alpha(f_\#(\sigma)) = \alpha(f \circ \sigma)$.
Then f^* is a cochain map. i.e. $d^* f^\# = f^\# d^*$.

Definition 9.7 (Chain homotopies): If C, C' are cochain complexes, $f, g : C \to C'$ are chain maps, then they are cochain homotopic if $f - g = d^*h + hd^*$.

Lemma 9.1:
If $f \sim g$ then $f^* = g^*$.
If $f, g : C \to C'$ are chain complexes, $f \sim g$ via h, the

$$f^*, g^* : \hom(C'; N) \to \mathrm{Hom}(C, N)$$

are cochain $\sim$ via h^*.

Remark 10 (Eilenberg Steenrod): Recall Eilenberg Steenrod about contravariant functors.

- $f_0, f_1 : (X, A) \to (Y, B)$ and $f_0 \sim f_1$ as map of pairs, then $f_0^* = f_1^* : H^*(Y, B) \to H^*(X, A)$
- Get a LES of pairs except it goes up and in opposite direction
- You also get excision except it is upper star
- Dimension: $H^*(\{\bullet\}, G) = G, * = 0, 0$ o.w.

Theorem 9.2:
Any functor satisfying the above Eilenberg Steenrod is given by $H^{\text{cell}}(X; G)$ when X is a finite cell CX.

9.2 Ext and Universal coefficient theorems

Definition 9.8 (Ext): If M, N are $R-$modules,

$$\mathrm{Ext}^i(M, N) = H^i(\mathrm{Hom}(A, N))$$

where A is a free resolution of M.

Theorem 9.3:
Given a X finite cell cx, you can decompose H^k into Ext and Hom.
You can also use $H_k(X)$ to split up into the freely generated parts and the finite parts. The rank of the freely generated part is the Betti number.

Lemma 9.4 (Pairing):
If C is a chain CX over R, then there is a bilinear pairing $\langle , \rangle : \text{Hom}(C_k; N) \times C_k \to N, \langle a, c\rangle = a(c)$ this descends to a pairing

$$H^k(\text{Hom}(C, N)) \times H_k(C) \to N$$

$$\langle [\alpha], [c]\rangle \to \langle \alpha, c\rangle \to \alpha(c)$$

9.3 Week 5 Lecture 3

9.4 Cup product

Definition 9.9 (Cup product):
If $a \in C^k(X; R), b \in C^l(X, R)$ then

$$a \cup b \in C^{k+l}(X; R)$$

is given by

$$a \cup b = a(\sigma \circ F_{0 \dots j})b(\sigma \circ F_{k \dots k+1})$$

Check that this definition makes sense!

Lemma 9.5 (Cup product makes ring):
$\cup$ makes $C^*(X; R)$ into a commutative ring with $1 \in C^0(X; R)$. Think of $\alpha \cup \beta(\sigma)$ as $\alpha(\sigma)\beta(\sigma)$ where we just need to bump up the degree to $a + b$.

Lemma 9.6 (Leibniz Rule):
If $\alpha \in C^k, \beta \in C^l$, then

$$d^*(\alpha \cup \beta) = (d^*\alpha) \cup \beta + (-1)^k \alpha \cup (d^*\beta)$$

Corollary 9.7 (Cup product descends to cohomology):
$\cup$ descends to a map

$$\cup : H^k(X; R) \times H^l(X; R) \to H^{l+k}(X; R)$$

$$[\alpha] \times [\beta] \mapsto [\alpha \cup \beta]$$

This makes $H^*(X; R)$ into a ring with $[1] = 1$.

Remark: notes that cup products make both cochain complexes and cohomologies into a ring

Proposition 9.8 (A very important prop):
If $f : X \to Y$ then $f^* : H^*(Y; R) \to H^*(X; R)$ is a ring homomorphism.
i.e.

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

Proposition 9.9 (A hard one):
$H^*(X;R)$ is graded commutative. This mean

$$a \cup b = (-1)^{|a||b|} b \cup a$$

where $|a| = k$ if a is an element of $H^k(X;R)$. Warning: this is only true for cohomology classes but not true for cochains in general

See Dexter page 48

Theorem 9.10:
If $r_j : C_j(X) \to C_j(X)$ defined based on flipping the corners of the simplices around, then $r_* : C_*(X) \to C_*(X)$ is a chain map and $r \sim 1_{C_*(X)}$.

10 Week 6

10.1 Week 6 lecture 1

Remark 11: Using half a class to prove the theorem about the simplex-flipping function r and used it to show graded commutativity.

Proposition 10.1 (Some properties about pairs):

- if $\alpha \in C^k(X, A)$ and $\beta \in C^l(X)$ then $\alpha \cup \beta \in C^*(X, A)$.
- $\cup$ defines a map
$$H^*(X, A) \times H^*(X) \to H^*(X, A)$$
$$(\alpha, \beta) \mapsto \alpha \cup \beta$$
- more generally, $\cup$ defines a map
$$H^*(X, A) \times H^*(X, B) \to H^*(X, A \cup B)$$
this is consequence of subdivision lemma.
- $$H^*(X \coprod Y) \cong H^*(X) \oplus H^*(Y)$$

11 Week 6 Lecture 2

Definition 11.1 (Exterior products): Setup: (X, A) is a pair of spaces. Y is a space.

$$\pi_1 : (X \times Y, A \times Y) \to (X, A)$$

$$(x, y) \mapsto x$$

$$\pi_2 : X \times Y \mapsto Y$$
$$(x, y) \mapsto y$$

Then, if $a \in H^k(X, A), b \in H^l(Y)$ then their exterior product $a \times b = \pi_1^*(a) \cup \pi_2^*(b) \in H^{k+l}(X \times Y, A \times Y)$.

Proposition 11.1 (Some observations about exterior product):

1. $$H^*(X, A) \times H^*(Y) \to H^*(X \times Y, A \times Y)$$
   $$(a, b) \mapsto a \times b$$
   is bilinear so it extends to
   $$H^*(X, A) \bigotimes H^*(Y) \to H^*(X \times Y, A \times Y)$$
   $$a \otimes b \mapsto a \times b$$
2. $$(a_1 \times b_1) \cup (a_2 \times b_2) = (-1)^{|a||b|}(a_1 \cup a_2) \cup (b_1 \cup b_2)$$

Theorem 11.2 (A quite big theorem):
If $H^*(Y; R)$ is free over R, then

$$\Phi : H^*(X, A; R) \otimes H^*(Y; R) \to H^*(X \times Y, A \times Y; R)$$

is an isomorphism.
It being free is very important. Note that if R is a field thenit is always free.
There are two consequences

- If the object is free then we can compute $H^*(X \times Y; R)$ from $H^*(X; R), H^*(Y; R)$.
- it tells us the ring structure on $H^*(X \times Y; R)$.

Corollary 11.3 (Help us identify another set of spaces):
Although $S^2 \times S^2$ have the same H_* as $S^2 \vee S^2 \vee S^4$, but they still have different structures as rings. So they are not homeomorphic.

12 Week 6 lecture 3

Theorem 12.1:
If X is an fcc, $\Phi : \underline{h}(x) \cong \overline{h}(x)$ is an iso.

Lemma 12.2:
Φ commutes with induced maps and δ-maps in LES of a pair.

Together these prove the big theorem in six step. This is a quite important proof!

13 Vector bundles

Definition 13.1 (Vector bundle): An $n-$diml real vector bundle (B, E, π) respectively are bas space, total space, and projection from total to base such that

- $\pi^{-1}(b)$ is a real $n-$diml vector space for each $b \in B$.
- there is an open cover $U_\alpha, \alpha \in A$ of B and maps $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ such that the following commutes.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{f_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ U_\alpha & \xrightarrow{id_{U_\alpha}} & U_\alpha \end{array}$$

  and $\pi_2 \circ f_\alpha : \pi^{-1}(b) \to \mathbb{R}^n$ is an isomorphism of vector spaces for all $b \in U_\alpha$. The f_α are localizations.

Similarly there is a complex $n-$diml vector bundle.
A morphism of vector bundles $f : (E, B, \pi) \to (E', B', \pi')$ is a commuting square

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[f_B]{} & B' \end{array}$$

such that $f_E \mid_{\pi^{-1}(B)}: \pi^{-1}(b) \to (\pi')^{-1}(f(b))$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$. But note that the fibres can have different dimensions.
E is a submodule of E' if there's an injective morphism

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[1_B]{} & B' \end{array}$$

i.e. $\pi^{-1}(b)$ is a linear subspace of $(\pi')^{-1}(b)$.

14 Week 7

14.1 Week 7 lecture 1

Definition 14.1 (A list of definitions):

- sections of a vector bundle E.
- a non-vanishing section
- Trivial bundle
- the mobius bundle
- the tautological bundle

Proposition 14.1 (Trivial vector bundle):
E is trivial $\iff$ there exists sections $s_1, \dots s_n : B \to E$ such that $\{s_1(b), \dots, s_n(b)\}$ is a basis for $\pi^{-1}(b)$ for all $b \in B$.

Definition 14.2 (Pullbacks of r-vector bundle): If $\pi : E \to B$ is an $n-$diml real vector bundle. and $g : B' \to B$ continuous, then

$$g^*(E) = \{(b', b, v) \in B' \times B \times E \mid g(b') = \pi(v) = b\}$$

where

$$\pi_g : g^*(E) \to B'$$
$$(b', b, v) \to b'$$

and

$$\pi_g^{-1}(b') = \{(b', g(b), v) \mid \pi(v) = g(b) = \pi^{-1}(g(b))\}$$

is a vector space.
If $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a local trivialization for E.
Let $V_\alpha = g^{-1}U_\alpha$.

$$f'_\alpha : \pi_g^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^n$$
$$(b', b, v) \mapsto (b', \pi_2(f_\alpha(v)))$$

is a local triv for $g^*(E)$.
So $g^*(E)$ is the pullback of E by g.

Lemma 14.2:

$$(g \circ f)^*(E) = f^*(g^*(E))$$

Definition 14.3 (Restriction): If $A \subseteq B$, $i : A \hookrightarrow B$ is the inclusion, then $E\mid_A := i^*(E)$ is the restriction of E to A.
If $s : B \to E$ is a non-vanishing section then $g^*s : B' \to g^*(E)$, $b' \mapsto (b', f(b), s(f(b)))$ is a nonvanishing section of $g^*(E)$.

Definition 14.4 (Products and sums): If $\pi : E \to B$, $\pi' : E' \to B'$ are r-vector bundles of dimension n, n', then their product $\pi \times \pi' : E \times E' \to B \times B'$ is a vector space of dimension $n' \times n$. The local trivializations are also defined similarly.
If $B = B'$, then $E \oplus E' = \Delta^*(E \times E') \to B$ where $\Delta : B \to B \times B, b \mapsto (b, b)$ is the whitney sum of E and E'.

Definition 14.5 (Partition of unity):

- Support of a function $\phi : B \to \mathbb{R}$
- partition of unity has
  1. $\in [0, 1]$
  2. indices such that $\phi_i(b) \neq 0$ is finite for all b
  3. support of all ϕ_i is in a single cover
  4. $\sum_i \phi_i(b) = 1$ for all $b \in B$.

A space B admits a PoU if for every open cover $U = \{U_\alpha \mid \alpha \in B\}$, there is a partition of unity subordinate to U. If B is cpt then B admits a PoU.
Compact Hausdorff spaces, metrisable spaces, manifolds, all admit partitions of unity.
In general, a space B admits partitions of unity if B is paracompact Hausdorff.

Theorem 14.3:
Suppose B admits a PoU and $\pi : E \to B \times I$ is a real VB. Then

$$E\mid_{B\times 0} \simeq E\mid_{B\times 1}$$

14.2 Week 7 Lec 2

Skipped

14.3 Week 7 Lec 3

Lemma 14.4:
If $E\mid_{B\times[0,1/2]}$ and $E\mid_{B\times[1/2,1]}$ are trivial, then E is trivial.

Lemma 14.5:
For each $b \in B$, b has an open neighbourhood U_b such that $E\mid_{U_b \times I}$ is trivial.

These two lemmas help prove the big theorem about the PoU implying homotopic equivalence.

Corollary 14.6:
Suppose that $\pi : E \to B$ is a vector bundle, $g_0, g_1 : B' \to B$, $g_0 \sim g_1$ via some $h : B' \times I \to B$ and that B' admits a PoU. Then

$$g_0^*(E) = h^*(E) \mid_{B' \times 0} \simeq h^*(E) \mid_{B' \times 1} = g_1^*(E)$$

Corollary 14.7:
If B is contractible and admits a PoU, then every VB $\pi : E \to B$ is trivial.

14.4 Riemannian metrics:

Definition 14.6 (Riemannian metric): Suppose $\pi : E \to B$ is a real VB (resp. complex VB). A Riemannian (resp. Hermitian) metric on E is a continuous map

$$g : E \oplus E \to \mathbb{R}$$

(resp. $\to \mathbb{C}$) such that

$$g \mid_{\pi^{-1}(E \oplus E)}$$

is an inner product (resp. a Hermitian inner prod)

$$\pi_{E \oplus E}^{-1}(b) = \pi^{-1}(b) \times \pi^{-1}(b)$$

Definition 14.7 (Unit disk, unit sphere bundles): Suppose E is a VB with Riemannian metric g. The the unit disk, and the unit bundles of E are given by

$$S_g(E) = \{v \in E \mid \langle v, v\rangle = 1\}$$

$$\pi : S_g(E) \to B, \pi^{-1}(b) \simeq S^{-1}$$

$$D_g(E) = \{v \in E \mid \langle v, v\rangle \leq 1\}$$

$$\pi : D_s(E) \to B, \pi^{-1}(b) \simeq D^n$$

Proposition 14.8:
If B admits PoU, $\pi : E \to B$ is a real VB, then E has a R-metric.

Definition 14.8 (The R-Thom class):
Given vector bundle, let $i_b : E_b \hookrightarrow E$ be the inclusion and $s_0 : B \to E$ be the 0 section. Define $E^{\#} = E \setminus Im(S_0)$ and $E_b^{\#} = E_b \setminus 0$.
Then $u \in H^n(E, E^{\#}; R)$ is an R- Thom class for E if $i_b^*(u)$ generates $H^*(E_b, E_b^{\#}; R)$ for all $b \in B$.

14.5 Week 8 Lec 1

Proposition 14.9 (Pullbacks):
If $f : B' \to B$, then there is a morphism of vector bundles.

$$\begin{array}{ccc} f^*(E) & \xrightarrow{F} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

Lemma 14.10 (Thom class behaves naturally under pullback):
If U is a $R-$ Thom Class for E, then $F^*(U)$ is an $R-$ Thom class for $f^*(E)$.

Lemma 14.11:
Suppose $B = B_1 \cup B_2$, $U \in H^n(E, E^\#)$. $i_k : B_k \to B$ is the inclusion. Then if $i_1^*(U)$ are thom class of $E\mid_{B_1}$ and $i_2^*(U)$ are Thom class of $E\mid_{B_2}$. Then U is a Thom class for E.

Theorem 14.12 (Quite important: the Thom Isomorphism):
If $\pi : E \to B$ is an n-dimensional $r-$vector bundle, then

- E has a unique $\mathbb{Z}/2$ Thom Class.
- If E has an $R-$Thom class U, the map

  $$\Phi : H^*(B; R) \to H^{*+n}(E, E^\#; R)$$

  is an isomorphism

  $$a \mapsto \pi^*(a) \cup U$$

14.6 The Gysin sequence

Definition 14.9 (The Euler class): If $\pi : E \to B$ is an $R-$oriented $n-$diml real vector bundle with Thom class U, then its Euler class is

$$e(E) = s_0^* j^*(U) \in H^n(B)$$

You can see this clearly on a commutative diagram (LES ladder of $(E, E^\#)$)

Theorem 14.13 (Gysin Sequence):
There is an LES:

$$\longrightarrow H^{*-n}(B) \xrightarrow{\alpha} H^*(B) \xrightarrow{\pi^*} H^*(\mathbb{S}(E)) \longrightarrow H^{*-n+1}(B) \longrightarrow$$

where $\alpha(a) = \alpha \cup e(E)$

14.7 Week 8 Lec 2

Proposition 14.14 (Properties of e):
Suppose that E is as above. Then

- $f : B' \to B$, then $f^*(E)$ is oriented and $e(f^*(E)) = f^*(e(E))$
- If E is trivial and $n > 0$, then $e(E) = 0$.
- $e(E_1 \oplus E_2) = e(E_1) \oplus e(E_2)$
- If E has a non-vanishing section then $e(E) = 0$.

Theorem 14.15:

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \simeq \mathbb{Z}/2[x]/(x^{n+1})$$

where $x = e(T_{\mathbb{RP}^n}) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$

Corollary 14.16:
$\pi_3(S^2) \neq 0$

Remark 12: Four remarks on orientability and the coefficients.

15 Manifolds

Definition 15.1 (n-manifold): An $n-$manifold is a 2nd countable Hausdorff space M with an open cover $\{U_\alpha \mid \alpha \in A\}$ and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$.
The transition functions $\psi_{\alpha\beta} : \phi_\alpha \circ \phi_\beta : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are homeomorphisms. M is smooth if ϕ_αs can be chosen so $\psi_{\alpha\beta}$ are diffeomorphisms.

Definition 15.2 (Fundamental class): We write $(M \mid A) = (M, M - A)$. Then if $B \subset A$ then $i : (M \mid A) \to (M \mid B)$ is inclusion of pairs. if $w \in H_*(M \mid A), w \mid_B = i_*(w)$.

Definition 15.3 (R-fundamental class): An $R-$fundamental class for $(M \mid A)$ is $w \in H_n(M \mid A; R)$ such that $w \mid_X$ generate $H_n(M \mid_X)$ for all $x \in A$. It's an analogue of Thom Class.

Theorem 15.1:
If $A \subset M$ is compact, $(M \mid A)$ has unique $\mathbb{Z}/2$ fundamental class. We are most intersted in the case when M is compact/closed. A fundamental class for $(M \mid M) = (M, \varnothing)$ will be written as $[M] \in H_n(M)$.

Proposition 15.2:
M is orientable if it has a $\mathbb{Z}$-fundamental class. M is orientable iff TM is orientable.

15.1 Week 8 Lec 3

Definition 15.4: $N \subset M$ is a $k-$dimensional smooth submanifold of an n-manifold M if for every $x \in N$, there is a smooth chat

$$\phi_x : U_x \to \mathbb{R}^n$$

such that

$$\phi_x(U_x \cap N) \to \mathbb{R}^k \times 0 \subset \mathbb{R}^n$$

If $N \subset M$ is a smooth submanifold then $TN \subset T_{M|N}$ is a subbundle.
Also $N \subset M$ is a smooth submani.

Definition 15.5: $V_{M|N} = TN^{\perp} \subset TM\mid_N$ us the normal bundle of N in M. So $TM\mid_N = V_{M|N} \oplus TN$.

Theorem 15.3 (Tabular neighbourhood theorem):
If $N \subset M$ is a closed smooth submanifold. There is an open $V \subset M, N \subset V$ with $(V, N) \simeq (V_{M|N}, s_0 V_{M|N})$

Lemma 15.4:
Suppose $E = E_1 \oplus E_2$ is orientable, then E_1 is orientable $\iff$ E_2 is orientable.

Proposition 15.5:
M is orientable $\iff$ TM is orientable.

Corollary 15.6:
If M is orientable, $N \hookrightarrow M$ is a closed smooth submani. Then M is orientable $\iff$ $V_{M|N}$ is orientable.

16 Poincaré duality

Now we work in coefficients in $\mathbb{F}$.
Note that $H^k(X) \simeq \text{Hom}(H_k(x), \mathbb{F})$. write $\langle a, \phi(\alpha)\rangle = \alpha(a)$. If $a \in H^k(X), a \cup \cdot : H^l(x) \to H^{k+l}(x)$.

Definition 16.1 (Cap product): $\cdot \cap a : H_{l+k}(x) \to H_l(x)$ is the dual of the above. $\langle b, x \cap a\rangle = \langle a \cup b, x\rangle$

Definition 16.2 (Intersection paring): Suppose M is an F-oriented n-manifold with $fund[M] \in H_n(M)$. The intersection pairing $(\cdot,\cdot) : H^k(M) \times H^{n-k}(M) \to \mathbb{F}$ is the bilinear pairing given by

$$(a, b) = \langle a \cup b, [M]\rangle$$

satisfying graded commutativity.
If $a \in H^k(M)$, $(a, \cdot) \in \mathrm{Hom}(H^{n-k}, \mathbb{F})$

Definition 16.3 (Algebraic Poincare dual): THe algebraic Poincare dual of a is

$$PD(a) = \phi((a, \cdot)) = [M] \cap a$$

so

$$\langle b, PD(a)\rangle = (a, b) = \langle a \cup b, [M]\rangle$$

Now we think about the Geometric poincare duality.

Theorem 16.1:
If M is a connected n-manifold. The map $H_n(M) \to H_n(M \mid x) = H_n(M, M - x) \simeq \mathbb{F}$ is injective.

Definition 16.4: $U_{M|N} = k^{-1}U$ is the orientation on $V_{M|N}$ induced by $[N], [M]$ it satisfies

$$\langle U_{M|N} \cup \pi^*[N]^*, i_*^{-1} j_*[M]\rangle = 1$$

Definition 16.5:

$$pd(N) = j^*((i^*)^{-1}(U_{M|N})) \in H^{n-k}(M)$$

is the geometry poincare dual of N.

Proposition 16.2:
if $a \in H^k(M)$ $PD(pd(N)) = i_*(N)$

Note that the algebraic poincare dual is PD and the grometric is pd.

Lemma 16.3:
$i : V \to M$, $i^*(a) = \langle a, i_*[N]\rangle \pi^*[N]^*$

16.1 Week 8 lec 4

Recall that $PD : H^k(M) \to H_{n-k}(M)$. This is given by

$$\langle b, PD(a)\rangle = (a, b) = \langle a \cup b, [M]\rangle$$

We have the identity:

$$\langle a \times b, \alpha \times \beta \rangle = \langle a, \alpha \rangle \langle b, \beta \rangle$$

Theorem 16.4 (Poincare duality):
PD is an iso.

Proof: Needs 3 lemmas and 3 props. It gives two corollaries. □

16.2 Intersection pairing on homology

Definition 16.6: Transverse submanifolds yields 4 different properties. Also $[N_1][N_2]$ product of two transverse submanifolds.

Proposition 16.5:
Two propositions about the geometric PD of N_is.

Corollary 16.6:
$\langle e(TM), [M] \rangle = \chi(M)$